

## MATH 20B HW8 SOLUTIONS

- Section 10.3: 3, 10, 13, 16, 22, 27, 32, 37, 47, 54, 62
  - Section 10.4: 6, 7, 9, 11, 24, 25, 28
  - Section 10.5: 3, 6, 10, 17, 21, 23, 43, 44, 50, 51
  - Section 10.6: 8, 11, 17, 23, 29, 35, 37
- $\equiv$

In exercises 3, 10, and 13, use the Integral Test to determine whether the infinite series is convergent.

10.3.3

$$\sum_{n=1}^{\infty} n^{-1/3}$$

Let  $f(x) = x^{-1/3}$ . Recall that from Thm. 1 on p. 468,  
 $\int_1^{\infty} f(x) dx = \infty$  (diverges)

Therefore  $\sum_{n=1}^{\infty} n^{-1/3}$  diverges by the Integral Test.

10.3.10

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Let  $f(x) = \frac{1}{x(\ln x)^2}$ , that letting  $u = \ln x$  gives

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{x=2}^{x=\infty} \frac{1}{u^2} du$$

$$= \left[ -\frac{1}{u} \right]_{x=2}^{x=\infty}$$

$$= \lim_{b \rightarrow \infty} \left( -\frac{1}{\ln(b)} + \frac{1}{\ln(2)} \right) = \frac{1}{\ln 2} \rightarrow \text{converges}$$

Hence the given series converges by the Integral Test.

10.3.13

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

Letting  $f(x) = \frac{\ln x}{x^2}$ ,  
 $\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{\ln x}{x^2} dx$

$$u = \ln x \quad v' = \frac{1}{x^2}$$

$$u' = \frac{1}{x} \quad v = -\frac{1}{x}$$

$$\begin{aligned}\therefore \int_1^{\infty} f(x) dx &= -\frac{\ln x}{x} \Big|_1^{\infty} + \int_1^{\infty} \frac{1}{x^2} dx \\&= -\lim_{b \rightarrow \infty} \left[ -\frac{\ln x}{x} - \frac{1}{x} \right]_1^b \\&= \lim_{b \rightarrow \infty} \left[ \left( -\frac{\ln b}{b} - \frac{1}{b} \right) - \left( -\frac{\ln 1}{1} - \frac{1}{1} \right) \right] \\&= 1 \rightarrow \text{converges}\end{aligned}$$

Therefore by the Integral Test, the given series converges also.

10.3.16

Show that  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$  diverges by comparing with  $\sum_{n=2}^{\infty} n^{-1}$ .

For all  $n \geq 2$ , we have that

$$n^2 - 3 \leq n^2$$

$$\therefore \sqrt{n^2-3} \leq \sqrt{n^2} = n$$

Because both  $\sqrt{n^2-3}$  and  $n$  are positive for  $n \geq 2$ , we can take the inverse of both sides as long as we flip the inequality:

$$\frac{1}{\sqrt{n^2-3}} \geq \frac{1}{n}$$

Therefore we can apply the Comparison Test because since  $\sum_{n=2}^{\infty} \frac{1}{n}$  diverges, so does  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ .

10.3.21

In Exercises 22 and 27, use the Comparison Test to determine whether the infinite series is convergent.

10.3.22

$$\sum_{n=4}^{\infty} \frac{\sqrt{n}}{n-3}$$

Just by looking at this series, we'd like to compare it with  $\frac{1}{\sqrt{n}}$  since we can roughly divide numerator and denominator by  $\sqrt{n}$ . So note that

$$\frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n}$$

At the same time  $n \geq n-3$ , so  $\frac{1}{n} \leq \frac{1}{n-3}$  (for  $n \geq 4$ ), so multiplying by  $\sqrt{n}$  on each side gives

$$\left( \frac{1}{\sqrt{n}} = \right) \frac{\sqrt{n}}{n} \leq \frac{\sqrt{n}}{n-3}$$

Thus the given series diverges by comparison with  $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}}$ .

10.3.27

$$\sum_{m=1}^{\infty} \frac{4}{m! + 4^m}$$

We can overcomplicate this by trying to use both  $m!$  and  $4^m$ . Instead let's note that  $m! \geq 0$  for all  $m \geq 1$ , so for all  $m \geq 1$

$$\frac{4}{m! + 4^m} \leq \frac{4}{4^m} = \frac{1}{4^{m-1}} = \left(\frac{1}{4}\right)^{m-1}$$

Hence we have that the given series must converge by comparison with

$$\sum_{m=1}^{\infty} \left(\frac{1}{4}\right)^{m-1} = \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m$$

(which is a convergent geometric series).

10.3.32 Show that  $\sum_{n=1}^{\infty} \sin(1/n^2)$  is a positive convergent series.

We know that for all  $n \geq 1$ ,  $0 < 1/n^2 \leq 1$ . But  $\sin x$  is positive on  $(0, 1]$  (think unit circle), so  $\sum_{n=1}^{\infty} \sin(1/n^2)$  is a positive series.

As for convergence, for  $x \geq 0$  we are given that  $\sin x \leq x$ , so that  $\sin(1/n^2) \leq 1/n^2$ . But we know that  $\sum_{n=1}^{\infty} 1/n^2$  is convergent, so that by the Comparison Test so is  $\sum_{n=1}^{\infty} \sin(1/n^2)$ .

In Exercise 37, use the Limit Comparison Test to prove convergence or divergence of the infinite series.

10.3.37

$$\sum_{n=3}^{\infty} \frac{n^3}{\sqrt{n^4 - 2n^2 + 1}}$$

In the long run, the terms of this series are going to "behave like"  $\frac{n^3}{n^2} = n$ . So we let  $a_n = \frac{n^3}{\sqrt{n^4 - 2n^2 + 1}}$  and  $b_n = n$ :

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n^3}{\sqrt{n^4 - 2n^2 + 1}}}{n} \cdot \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{\sqrt{n^4 - 2n^2 + 1}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} \cdot n^2}{\frac{1}{n^2} \sqrt{n^4 - 2n^2 + 1}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 - \frac{2}{n^2} + \frac{1}{n^4}}} \\ &= 1\end{aligned}$$

Therefore since  $\sum_{n=3}^{\infty} b_n = \sum_{n=3}^{\infty} n$  diverges, so does the given series.

10.3. 47 For which  $a$  does  $\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$  converge?

Let's apply the Integral Test. Let  
 $f(x) = \frac{1}{x^a \ln x}$

If  $a=1$  then

$$\begin{aligned}\int_2^{\infty} f(x) dx &= \int_2^{\infty} \frac{1}{x \ln x} dx \\ &= \lim_{b \rightarrow \infty} \int_{x=2}^{x=b} \frac{1}{u} du \quad (u = \ln x) \\ &= \lim_{b \rightarrow \infty} [\ln|u|]_{x=2}^{x=b} \\ &= \lim_{b \rightarrow \infty} [\ln(\ln x)]_{2}^{b} \\ &= \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty \rightarrow \text{DIVERGES}\end{aligned}$$

If  $a \neq 1$ , then

$$\begin{aligned}\int_2^{\infty} f(x) dx &= \int_2^{\infty} \frac{1}{x^a \ln x} dx \\ u &= \frac{1}{\ln x} & v &= \frac{1}{x^a} = x^{-a} \\ u' &= -\frac{1}{x \ln x} & v' &= \frac{x^{-a+1}}{-a+1} \\ \therefore \int_2^{\infty} \frac{dx}{x^a \ln x} &= \lim_{b \rightarrow \infty} \left( \frac{x^{-a+1}}{(-a+1) \ln x} \right]_2^b - \int_2^{\infty} \frac{x^{-a}}{(-a+1) \ln x} dx \\ \therefore \int_2^{\infty} \frac{dx}{x^a \ln x} + \frac{1}{-a+1} \cdot \int_2^{\infty} \frac{dx}{x^a \ln x} &= \lim_{b \rightarrow \infty} \left( \frac{x^{-a+1}}{(-a+1) \ln x} \right]_2^b \\ \therefore \left( \frac{-a+2}{-a+1} \right) \int_2^{\infty} \frac{dx}{x^a \ln x} &= \lim_{b \rightarrow \infty} \left( \frac{b^{-a+1}}{(-a+1) \ln b} - \frac{2^{-a+1}}{(-a+1) \ln 2} \right) \\ \therefore \int_2^{\infty} \frac{dx}{x^a \ln x} &= \frac{1-a}{2-a} \left[ \lim_{b \rightarrow \infty} \left( \frac{1}{(1-a)b^{a-1} \ln b} \right) - \frac{1}{(1-a)2^{a-1} \ln 2} \right]\end{aligned}$$

The limit on the right diverges if  $a < 1$  and converges if  $a \geq 1$ . Hence by the Integral Test combined with what we have above, our series diverges for  $a \leq 1$  and converges for  $a > 1$ .

In Exercises 54 and 62, determine convergence or divergence using any method covered so far.

10.3.54

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$$

It would be nice to use the Comparison Test by comparing with  $\sum 1/n^2$ , but we would need that for all  $n$

$$\frac{1}{n^2 + \sin n} \leq \frac{1}{n^2}$$

which is not necessarily true because  $\sin n < 0$  for some values of  $n$ . So instead we use the Limit Comparison Test.

So let  $a_n = 1/(n^2 + \sin n)$  and  $b_n = 1/n^2$ , and observe that

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1/(n^2 + \sin n)}{1/n^2} \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \sin n} \\ &= \lim_{n \rightarrow \infty} \frac{(1/n^2) \cdot n^2}{(1/n^2)(n^2 + \sin n)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{\sin n}{n^2}} \\ &= 1\end{aligned}$$

Therefore because  $\sum b_n = \sum 1/n^2$  converges, so does the given series.

10.3.62

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n - n}$$

We would like to compare this series to  $\sum 1/\ln n$ , which we know diverges by Exercise 47 above. So note that for  $n \geq 2$ ,

$$0 \leq n \ln n - n \leq n \ln n$$

$$\therefore \frac{1}{n \ln n - n} \geq \frac{1}{n \ln n}$$

Hence by the Comparison Test, since  $\sum_{n=2}^{\infty} 1/\ln n$  diverges so does  $\sum_{n=2}^{\infty} 1/(n \ln n - n)$ .

In Exercises 6, 7, 9, and 11, determine whether the series converges absolutely, conditionally, or not at all.

10.4.6

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

Observe that for all  $n$ ,  $|\sin n| \leq 1$ , so that

$$\left| \frac{\sin n}{n^2} \right| = \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

But we know that  $\sum_{n=1}^{\infty} 1/n^2$  converges, so by the Comparison Test  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  converges absolutely.

10.4.7

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

Note that for all  $n \geq 2$

$$\left| \frac{(-1)^{n+1}}{n \ln n} \right| = \frac{1}{n \ln n}$$

and since  $\sum \frac{1}{n \ln n}$  diverges (see Exer. 10.3.47)  
our series can't be absolutely convergent.

We can apply the Leibniz Test to determine that this series is conditionally convergent. Let

$$a_n = \frac{1}{n \ln n}$$

and observe that trivially  $\lim_{n \rightarrow \infty} a_n = 0$ . Secondly, to test that the terms are decreasing (we know they're positive) let

$$f(x) = \frac{1}{x \ln x}$$

$$\begin{aligned} \therefore f'(x) &= \frac{-1}{(x \ln x)^2} \cdot (\ln x + 1) \quad (\text{Chain Rule}) \\ &= -\frac{(\ln x + 1)}{(x \ln x)^2} < 0 \quad \text{for all } x \geq 2 \end{aligned}$$

Hence since  $f$  is decreasing we have  
 $a_2 \geq a_3 \geq a_4 \geq \dots$  and by the Leibniz Test  
we conclude that our given series  
converges conditionally.

10.4.9

$$\sum_{n=1}^{\infty} \frac{\sin n\pi}{\sqrt{n}}$$

Note that for all integers  $n \geq 1$ ,  $\sin n\pi = 0$   
so that

$$\sum_{n=1}^{\infty} \frac{\sin n\pi}{\sqrt{n}} = \sum_{n=1}^{\infty} 0 = 0$$

which is trivially absolutely convergent.

10.4.11

$$\sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2}$$

By an identical argument to that given in  
Exercise 10.4.6, this series converges absolutely.

In Exercises 24 and 25, determine convergence or divergence by any method.

10.4.24

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}$$

This says nothing about absolute convergence, so we can test convergence by the Leibniz Test. Let

$$a_n = \frac{1}{(2n+1)!}$$

and observe that clearly the  $a_n$  are positive with  $a_1 \geq a_2 \geq a_3 \geq \dots$  and  $\lim_{n \rightarrow \infty} a_n = 0$  so that by the Leibniz Test our series converges.

\* Note: It also converges absolutely, but it's a tricky argument given what we know up to this point and we don't need it for this problem.

10.4.25

$$\sum_{n=1}^{\infty} (-1)^n n e^{-n}$$

We would like to apply the Integral Test.  
To do so, however, we need positive terms. So set

$$a_n = |(-1)^n n e^{-n}| = n e^{-n}$$

and note that if  $\sum a_n$  converges then so does the given series by the theorem in the section. So we apply the Integral Test to  $a_n$ . Set

$$f(x) = x e^{-x}$$

and note that for the associated integral, integration by parts gives

$$\int_1^{\infty} x e^{-x} dx$$

$$u = x \quad v' = e^{-x}$$

$$u' = 1 \quad v = -e^{-x}$$

$$\begin{aligned} \therefore \int_1^{\infty} x e^{-x} dx &= -x e^{-x} \Big|_1^{\infty} + \int_1^{\infty} e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[ -x e^{-x} \Big|_1^b + \int_1^b e^{-x} dx \right] \\ &= \lim_{b \rightarrow \infty} \left[ -x e^{-x} \Big|_1^b - e^{-x} \Big|_1^b \right] \\ &= \lim_{b \rightarrow \infty} \left[ \left( -b e^{-b} - e^{-b} \right) - \left( -e^{-1} - e^{-1} \right) \right] \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{e^b} (-b - 1) + \frac{2}{e} \right] = \frac{2}{e} < \infty \end{aligned}$$

Hence our series converges (absolutely) by the Integral Test. (We could also use Leibniz + L'Hôpital)

10.4.28 The Leibniz Test cannot be applied to

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \dots$$

Why not? Show that it converges by another method.

This series can be written in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where the  $a_n$  are positive with  $\lim_{n \rightarrow \infty} a_n = 0$ ; however, we don't have  $a_1 \geq a_2 \geq a_3 \geq \dots$  because  $a_1 = \frac{1}{2}$ ,  $a_2 = \frac{1}{3}$ ,  $a_3 = \frac{1}{4}$ ,  $a_4 = \frac{1}{9}$ ,  $a_5 = \frac{1}{8}$  and  $\frac{1}{9} < \frac{1}{8}$ .

Instead, let's just write our series as

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{2^n} - \frac{1}{3^n} \right) &= \sum_{n=1}^{\infty} \left( \frac{3^n - 2^n}{2^n \cdot 3^n} \right) \\ &= \sum_{n=1}^{\infty} \left( \frac{3^n - 2^n}{6^n} \right) \end{aligned}$$

This is a series of positive terms with

$$\frac{3^n - 2^n}{6^n} \leq \frac{3^n}{6^n} = \left(\frac{3}{6}\right)^n = \left(\frac{1}{2}\right)^n$$

for all  $n$ . But  $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  is a convergent (geometric) series. So by the Comparison Test our series converges.

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In Exercises 3, 6, 10, and 17 apply the Ratio Test to determine convergence / divergence, or state that the Ratio Test is inconclusive.

10.5.3

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

$$a_n = \frac{(-1)^{n-1}}{n^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{(n+1)^{n+1}} \cdot \frac{n^n}{(-1)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n}{(n+1)^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{n+1}/n^n} \right| \quad (\text{Divide top, bottom by } n^n) \\ &= \lim_{n \rightarrow \infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n}{n^{n+1} + \dots + 1} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{n + \dots + 1/n^n} \right| \\ &= 0 < 1 \end{aligned}$$

Hence the given series converges.

10.5.6

$$\sum_{n=1}^{\infty} \frac{2^n}{n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| 2 \cdot \frac{n}{n+1} \right| \\ &= 2 > 1 \end{aligned}$$

Hence the given series diverges.

10.5.10

$$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e}{n+1} \right|$$

$$= 0 < 1$$

Hence the given series converges.

10.5.17

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n^2 + 2n + 1}{n^2} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n^2 + 2n + 1}{n^2} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+2)(2n+3)} \right|$$

$$= 1 \cdot 0 = 0 < 1$$

Hence the given series converges.

10.5.21 Show that  $\sum_{n=1}^{\infty} 2^n x^n$  converges if  $|x| < 1/2$ .

Let  $a_n = 2^n x^n$ , so that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} \cdot x^{n+1}}{2^n \cdot x^n} \right|$$

$$= \lim_{n \rightarrow \infty} |2x|$$

$$= |2x|$$

But our series will converge if  $|2x| < 1$ , or equivalently if  $|x| < 1/2$ .

10.5.23 Show that  $\sum_{n=1}^{\infty} r^n/n$  converges if  $|r| < 1$ .

Let  $a_n = r^n/n$ , so that

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{r^{n+1}}{n+1} \cdot \frac{n}{r^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \cdot \frac{r^{n+1}}{r^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \cdot \lim_{n \rightarrow \infty} \left| r \right| \\ &= 1 \cdot |r| = |r|\end{aligned}$$

So if  $|r| < 1$ , our series converges by the Ratio Test.

In Exercises 43, 44, 50, and 51, determine convergence or divergence by any method covered so far.

10.5.43  $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$

We will apply the Ratio Test. So

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{5^n}{5^{n+1}} \right| \\ &= 1 \cdot \frac{1}{5} = \frac{1}{5} < 1\end{aligned}$$

Hence our series converges.

10.5.44

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

We apply the Integral Test. So  $f(x) = \frac{1}{x(\ln x)^3} =$

$$\begin{aligned} \int_2^{\infty} f(x) dx &= \int_2^{\infty} \frac{1}{x(\ln x)^3} dx \\ &= \int_{x=2}^{\infty} \frac{du}{u^3} \quad (u = \ln x) \\ &= \left[ \frac{-1}{2u^2} \right]_{x=2}^{x=\infty} \\ &= \left[ \frac{-1}{2(\ln x)^2} \right]_2^{\infty} \\ &= \lim_{b \rightarrow \infty} \left[ \frac{-1}{2(\ln b)^2} + \frac{1}{2(\ln 2)^2} \right] \\ &= 0 + \frac{1}{2(\ln 2)^2} < \infty \end{aligned}$$

Hence  $\int_2^{\infty} f(x) dx$  converges and so does our series.

10.5.50

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

Let  $a_n = \frac{1}{\sqrt{n}}$ . Note that  $a_1 \geq a_2 \geq a_3 \geq \dots$ ,  $\lim a_n = 0$ , and the  $a_n$  are all positive, so by the Leibniz Test our series converges.

10.5.51

$$\sum_{n=1}^{\infty} \left( \frac{n}{n+12} \right)^n$$

Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &\Rightarrow \lim_{n \rightarrow \infty} \frac{n^n}{(n+12)^n} = \lim_{n \rightarrow \infty} \frac{n^n}{n^n + \dots + 12^n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \dots + 12^n/n^n} \quad (\text{division by } n^n) \\ &= 1 \end{aligned}$$

Hence because the limit of the terms is not zero,  
 the Divergence Test gives that the series diverges.

In Exercises 8, 11, 17, and 23, find the values  
 of  $x$  for which the following power  
 series converge.

10.6.8

$$\sum_{n=1}^{\infty} n(x-3)^n$$

We first find the radius of convergence  $R$ .

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| = 1 \Rightarrow R = 1$$

We then test convergence at  $x = 3 \pm 1 = 2, 4$ :

$$x=2: \sum_{n=1}^{\infty} n(2-3)^n = \sum_{n=1}^{\infty} n(-1)^n \rightarrow \text{diverges (Divergence Test)}$$

$$x=4: \sum_{n=1}^{\infty} n(4-3)^n = \sum_{n=1}^{\infty} n \rightarrow \text{diverges (Divergence Test)}$$

Hence our series converges for  $2 < x < 4$ .

10.6.11

$$\sum_{n=2}^{\infty} \frac{x^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} \cdot (x-0)^n$$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{\ln(n)}{\ln(n+1)} \right| = 1$$

We then test convergence at  $x = 0 \pm 1 = -1, 1$ :

$$x=-1: \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \rightarrow \text{converges (Leibniz Test)}$$

$$x=1: \sum_{n=2}^{\infty} \frac{(1)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n} \rightarrow \text{diverges (Compare with } \frac{1}{n})$$

Hence our series converges for  $-1 \leq x < 1$ .

10.6.17

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n = \sum_{n=0}^{\infty} \frac{n}{2^n} (x-0)^n$$

$$\begin{aligned}\frac{1}{R} &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| \cdot \lim_{n \rightarrow \infty} \left| \frac{2^n}{2^{n+1}} \right| \\ &= 1 \cdot \frac{1}{2} = \frac{1}{2} \Rightarrow R = 2\end{aligned}$$

Next we test convergence at endpoints  $x = 0 \pm 2 = -2, 2$ :

$$\begin{aligned}x = -2 : \sum_{n=0}^{\infty} \frac{n}{2^n} (-2)^n &= \sum_{n=0}^{\infty} n(-1)^n \rightarrow \text{diverges (Divergence Test)} \\ x = 2 : \sum_{n=0}^{\infty} \frac{n}{2^n} (2)^n &= \sum_{n=0}^{\infty} n \rightarrow \text{diverges (Divergence Test)}\end{aligned}$$

Hence our power series converges for  $-2 < x < 2$ .

10.6.23

$$\sum_{n=12}^{\infty} e^n (x-2)^n$$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{e^{n+1}}{e^n} \right| = \lim_{n \rightarrow \infty} |e| = e \Rightarrow R = \frac{1}{e}$$

Next we test convergence at  $x = 2 \pm 1/e = 2 - 1/e, 2 + 1/e$ :

$$\begin{aligned}x = 2 - \frac{1}{e} : \sum_{n=12}^{\infty} e^n \left(2 - \frac{1}{e} - 2\right)^n &= \sum_{n=12}^{\infty} e^n \left(\frac{-1}{e}\right)^n \\ &= \sum_{n=12}^{\infty} \frac{e^n}{e^n} (-1)^n \\ &= \sum_{n=12}^{\infty} (-1)^n \rightarrow \text{diverges (Divergence Test)}\end{aligned}$$

$$\begin{aligned}x = 2 + \frac{1}{e} : \sum_{n=12}^{\infty} e^n \left(2 + \frac{1}{e} - 2\right)^n &= \sum_{n=12}^{\infty} e^n \left(\frac{1}{e}\right)^n \\ &= \sum_{n=12}^{\infty} 1^n \rightarrow \text{diverges (Divergence Test)}\end{aligned}$$

Hence our series converges for

$$2 - \frac{1}{e} < x < 2 + \frac{1}{e}$$

10.6.29

Use Eq. (1) to expand the function in a power series with center  $c=0$  and determine the set of  $x$  for which the expansion is valid.

$$f(x) = \frac{1}{3-x}$$

Equation (1) says that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so we need to get  $f(x)$  in the form

$$a \cdot \frac{1}{1-g(x)}$$

so that we can write

$$f(x) = a \sum_{n=0}^{\infty} (g(x))^n$$

Abstract argument aside, this amounts to just multiplying numerator and denominator by  $\frac{1}{3}$ :

$$\begin{aligned} f(x) &= \frac{\frac{1}{3}}{\frac{1}{3}(3-x)} \\ &= \frac{\frac{1}{3}}{1 - \frac{x}{3}} \\ &= \frac{1}{3} \cdot \left( \frac{1}{1 - \frac{x}{3}} \right) \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^n \\ &= \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} \cdot x^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} (x-0)^n \end{aligned}$$

Next we find the radius of convergence:

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{1}{3^{n+1}} \cdot \frac{3^n}{1} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{3} \right| = \frac{1}{3} \Rightarrow R = 3$$

Finally we check convergence at  $x = 0 \pm 3 = -3, 3$ :

$$x = -3: \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} (-3)^n = \sum_{n=0}^{\infty} (-1)^n \rightarrow \text{diverges} \quad (\text{Divergence Test})$$

$$x = 3: \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} (3)^n = \frac{1}{3} \sum_{n=0}^{\infty} 1^n \rightarrow \text{diverges} \quad (\text{Divergence Test})$$

Hence our power series representation for  $f(x)$  converges for  $-3 < x < 3$ .

10.6.35 Use the equalities

$$\frac{1}{1-x} = \frac{1}{-3-(x-4)} = \frac{-1/3}{1+\left(\frac{x-4}{3}\right)}$$

to show that for  $|x-4| < 3$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{3^{n+1}}$$

By equation (1), we know that

$$\begin{aligned} \frac{-1/3}{1+\left(\frac{x-4}{3}\right)} &= -\frac{1}{3} \cdot \frac{1}{1-\left(-\left(\frac{x-4}{3}\right)\right)} = -\frac{1}{3} \sum_{n=0}^{\infty} \left(-\left(\frac{x-4}{3}\right)\right)^n \\ &= -\frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-4}{3}\right)^n \\ &= \sum_{n=0}^{\infty} (-1) \cdot (-1)^n \cdot \frac{1}{3} \cdot \frac{(x-4)^n}{3^n} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{(x-4)^n}{3^{n+1}} \end{aligned}$$

Secondly, we need to show that this converges for  $|x-4| < 3$  (or for  $1 < x < 7$ ). First we find the radius of convergence:

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}}{3^{n+2}} \cdot \frac{3^{n+1}}{(-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-1}{3} \right| = \frac{1}{3} \Rightarrow R = 3$$

Finally it remains to check the endpoints  $x = 4 \pm 3 = 1, 7$ :

$$x = 1: \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{3^n}{3^{n+1}} = \sum_{n=0}^{\infty} -\frac{1}{3} \rightarrow \text{diverges} \quad (\text{Divergence Test})$$

$$x = 7: \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{3^n}{3^{n+1}} = \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{1}{3} \rightarrow \text{diverges} \quad (\text{Divergence Test})$$

10.6.37

Use the method of Exercise 35 to expand  $\frac{1}{4-x}$  in power series with center  $c=5$ . Determine the set of  $x$  for which the expansion is valid.

Abstractly, what this means is that we need a power series for  $f(x) = \frac{1}{4-x}$  of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

To do this, we would like to use Equation (1) (as in the previous two exercises) to write

$$\frac{1}{4-x} = \frac{a}{1-g(x-5)}$$

So

$$\begin{aligned}\frac{1}{4-x} &= \frac{1}{4-5+x+5} \\ &= \frac{1}{-1-(x-5)} \\ &= \frac{1}{1+(x-5)} \\ &= \frac{1}{1-(-(x-5))} \\ &= \sum_{n=0}^{\infty} (-1)^n (x-5)^n\end{aligned}$$

Now, this has radius of convergence

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \right| = 1 \Rightarrow R = 1$$

so it remains to check the endpoints  $x = 5 \pm 1 = 4, 6$ :

$$x=4: \sum_{n=0}^{\infty} (-1)^n (4-5)^n = \sum_{n=0}^{\infty} (-1)^n (-1)^n = \sum_{n=0}^{\infty} 1^n \rightarrow \text{diverges}$$

$$x=6: \sum_{n=0}^{\infty} (-1)^n (6-5)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 1^n = \sum_{n=0}^{\infty} (-1)^n \rightarrow \text{diverges}$$

Hence our expansion of  $\frac{1}{4-x}$  is  $\{x : 4 < x < 6\}$ .

