

10.7.30

What is the Maclaurin series for  $f(x) = x^4 - 2x^2 + 3$ ?

The Maclaurin series for  $f(x)$  is just  $x^4 - 2x^2 + 3$ ; in fact, the Maclaurin series for any polynomial is just the polynomial itself (which holds for all  $x$ ). We can verify this if we'd like.

What is the Taylor series centered at  $c=2$ ?

$$f(x) = x^4 - 2x^2 + 3 \Rightarrow f(2) = 16 - 8 + 3 = 11$$

$$f'(x) = 4x^3 - 4x \Rightarrow f'(2) = 32 - 8 = 24$$

$$f''(x) = 12x^2 - 4 \Rightarrow f''(2) = 48 - 4 = 44$$

$$\vdots \quad \vdots \quad \vdots$$

## MATH 20B HW9 SOLUTIONS

- Section 10.7: 1, 5, 9, 20, 30, 31, 37, 55, 56
- Section 9.1: 2, 3, 6, 13, 17, 21, 34, 35, 43
- Section 9.2: 5, 11, 17, 22
- Section 9.4: 1, 6, 9

10.7.1

Write out the first four terms of the Maclaurin series of  $f(x)$  if

$$f(0) = 2, \quad f'(0) = 3, \quad f''(0) = 4, \quad f'''(0) = 12$$

In Exercises 31 and 37, find the Taylor series centered at  $c$ .

$$10.7.31 \quad f(x) = \frac{1}{x}, \quad c = 1$$

Method 1 :

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}$$

⋮

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \Rightarrow f^{(n)}(1) = (-1)^n n!$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} (x-1)^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \quad 0 < x < 2} \quad \left( \text{ROC} = 1, \text{divergence at } \begin{matrix} x=0, \\ x=2 \end{matrix} \right)$$

Method 2: Notice that plugging in  $x-1$  in place of  $x$  in  $\frac{1}{1+x}$  gives

$$\frac{1}{1+(x-1)} = \frac{1}{x}$$

so we can use the existing table:

$$\boxed{\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \quad |x-1| < 1}$$

10.7.37

$$f(x) = e^{3x}, \quad c = -1$$

Method 1:

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$f''(x) = 9e^{3x}$$

$$f'''(x) = 27e^{3x}$$

⋮

$$f^{(n)}(x) = 3^n e^{3x} \Rightarrow f^{(n)}(-1) = 3^n \cdot e^{-3}$$

$$\therefore e^{3x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} \cdot (x+1)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{3^n \cdot e^{-3}}{n!} (x+1)^n}$$

Check: ROC =  $\infty$ Method 2: From the representation of  $e^x$ , we have

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

which won't work since this isn't centered at  $c = -1$ . So instead we need  $3x = g(x+1)$  for some function  $g$ . But note that

$$3x = 3(x+1) - 3 \Rightarrow e^{3x} = e^{3(x+1)-3} = e^{-3} \cdot e^{3(x+1)}$$

$$\therefore e^{3x} = e^{-3} \cdot e^{3(x+1)} = \sum_{n=0}^{\infty} e^{-3} \cdot \frac{[3(x+1)]^n}{n!} \quad \text{All } x$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{\hat{e}^{-3} \cdot 3^n}{n!} (x+1)^n} \quad \text{All } x$$

In Exercises 55 and 56, express the integral as an infinite series.

10.7.55  $\int_0^x \frac{t - \sin t}{t} dt$ , for all  $x$

By FTC, differentiating w/r/t  $x$  yields

$$\begin{aligned}\frac{d}{dx} \left[ \int_0^x \frac{t - \sin t}{t} dt \right] &= \frac{x - \sin x}{x} = 1 - \frac{1}{x} \cdot \sin x \\ &= 1 - \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (\text{All } x) \\ &= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n+1)!}\end{aligned}$$

So we know that this is the derivative of our given function. Therefore we integrate both sides to obtain

$$\int_0^x \frac{t - \sin t}{t} dt = x - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)! \cdot (2n+1)} + C$$

To find  $C$ , note that

$$\begin{aligned}0 &= \int_0^0 \frac{t - \sin t}{t} dt = 0 - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0^{2n+1}}{(2n+1)! \cdot (2n+1)} + C \\ &= 0 - 0 + C \Rightarrow C = 0\end{aligned}$$

$$\boxed{\int_0^x \frac{t - \sin t}{t} dt = x - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)! \cdot (2n+1)}}$$

10.7.56

$$\int_0^x \ln(1+t^2) dt, \quad \text{for } |x| < 1$$

We proceed as in Exercise 55:

$$\frac{d}{dx} \left[ \int_0^x \ln(1+t^2) dt \right] = \ln(1+x^2)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$$

$$\therefore \int_0^x \ln(1+t^2) dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^{2n+1}}{n(2n+1)} + C$$

$$0 = \int_0^0 \ln(1+t^2) dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 0^{2n+1}}{n(2n+1)} + C$$

$$= 0 + C \Rightarrow C = 0$$

$$\therefore \boxed{\int_0^x \ln(1+t^2) dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot x^{2n+1}}{n(2n+1)}}$$

=

9.1.2 Which of the equations in Exercise 1 are linear?

(a)  $y' = x^2 \rightarrow \text{YES}$

(b)  $y'' = y^2 \rightarrow \text{NO} \quad (y^2 \text{ not a fn. of } x)$

(c)  $(y')^3 + yy' = \sin x \rightarrow \text{NO} \quad (y' \text{ multiplied by } y)$

(d)  $x^2 y' - e^x y = \sin y \rightarrow \text{NO} \quad (\sin y \text{ not a fn. of } x)$

(e)  $y'' + 3y' = \frac{y}{x} \rightarrow \text{YES} \quad (\text{equivalent to } y'' + 3y' - \frac{1}{x} \cdot y = 0)$

(f)  $yy' + x + y = 0 \rightarrow \text{NO} \quad (y' \text{ multiplied by } y)$

In Exercises 3 and 6, verify that the given function is a solution of the differential equation.

9.1.3

$$y' - 8x = 0, \quad y = 4x^2$$

$$y' = 8x$$

$$\therefore y' - 8x = 8x - 8x = 0 \quad \checkmark$$

(Note: Misprint in text)

9.1.6

$$y' + 4xy = 0, \quad y = 3e^{-2x^2}$$

$$y' = 3 \cdot (-4x) \cdot e^{-2x^2} \quad (\text{Chain Rule})$$

$$\begin{aligned} \therefore y' + 4xy &= 3 \cdot (-4x) \cdot e^{-2x^2} + 4x \cdot 3e^{-2x^2} \\ &= -12x e^{-2x^2} + 12x e^{-2x^2} \\ &= 0 \quad \checkmark \end{aligned}$$

In Exercises 13, 17, and 21, solve using separation of variables.

9.1.13

$$y' = xy^2$$

$$\frac{dy}{dx} = xy^2$$

$$\frac{dy}{y^2} = x dx$$

$$\therefore -\frac{1}{y} = \frac{x^2}{2} + C \Rightarrow \boxed{y = \frac{-1}{\frac{x^2}{2} + C}}$$

9.1.17

$$2 \frac{dy}{dx} + 6y + 4 = 0$$

$$\frac{dy}{dx} + 3y + 2 = 0$$

$$\frac{dy}{dx} = -3y - 2$$

$$\frac{dy}{3y+2} = -dx$$

$$\int \frac{dy}{3y+2} = \int -dx$$

$$\frac{1}{3} \ln |3y+2| = -x + C$$

$$\ln |3y+2| = -3x + 3C$$

$$e^{\ln |3y+2|} = e^{-3x+3C}$$

$$3y+2 = e^{3C} \cdot e^{-3x}$$

$$3y = e^{3C} \cdot e^{-3x} - 2$$

$$\therefore y = \frac{e^{3C}}{3} \cdot e^{-3x} - \frac{2}{3}$$

$$\boxed{y = C \cdot e^{-3x} - \frac{2}{3}}$$

(This is a great example of why we wait to bring out the capital - C at the very end)

9.1.21

$$y' = y^2(1-x^2)$$

$$\frac{dy}{dx} = y^2(1-x^2)$$

$$\frac{dy}{y^2} = (1-x^2)dx$$

$$\int \frac{dy}{y^2} = \int (1-x^2)dx$$

$$-\frac{1}{y} = x - \frac{x^3}{3} + C$$

$$\boxed{y = \frac{-1}{x - \frac{x^3}{3} + C}}$$

In Exercises 34 and 35, solve the initial value problem.

9.1.34

$$(1-t)\frac{dy}{dt} - y = 0, \quad y(2) = -4$$

$$(1-t)\frac{dy}{dt} = y$$

$$\int \frac{dy}{y} = \int \frac{-1}{t-1} dt$$

$$\ln|y| = -\ln|t-1| + C$$

$$e^{\ln|y|} = e^{-\ln|t-1|+C}$$

$$y = e^C \cdot e^{\ln(t-1)} = C \frac{1}{t-1}$$

$$\therefore y(2) = -4 = \frac{C}{2-1}$$

$$\therefore -4 = C$$

$$\boxed{y = \frac{-4}{t-1}}$$

9.1.35

$$\frac{dy}{dt} = ye^{-t}, \quad y(0) = 1$$

$$\frac{dy}{y} = e^{-t} dt$$

$$\int \frac{dy}{y} = \int e^{-t} dt$$

$$\ln|y| = -e^{-t} + C$$

$$e^{\ln|y|} = e^{-e^{-t} + C}$$

$$\therefore y = e^C \cdot e^{-e^{-t}} = Ce^{-e^{-t}}$$

$$y(0) = 1 = Ce^{-e^0} = Ce^{-1} = \frac{C}{e} \Rightarrow C = e$$

$$\boxed{\therefore y = e \cdot e^{-e^{-t}} = e^1 \cdot e^{-e^{-t}} = e^{1-e^{-t}}}$$

9.1.43 Find all values of  $a$  such that  $y = e^{ax}$  is a solution of

$$y'' + 2y' - 8y = 0$$

If  $y = e^{ax}$ , then

$$y' = ae^{ax}, \quad y'' = a^2 e^{ax}$$

$$\therefore y'' + 2y' - 8y = a^2 e^{ax} + 2ae^{ax} - 8e^{ax} = 0$$

$$(a^2 + 2a - 8)e^{ax} = 0$$

$$e^{ax} \neq 0 \text{ for all } x, \text{ so } a^2 + 2a - 8 = 0$$

$$(a+4)(a-2) = 0$$

$$\boxed{\therefore a = -4, a = 2}$$

(Note: this method in further mathematics courses will be generalized to solve all differential equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

It will involve complex numbers actually and they turn out to be very "useful" after all.)

9.2.5 A hot metal bar is submerged in a large reservoir of water whose temperature is  $60^{\circ}\text{F}$ . The temperature of the bar 20 s after submersion is  $100^{\circ}\text{F}$ . After 1 min, the temperature has cooled to  $80^{\circ}\text{F}$ .

(a) Determine the cooling constant  $k$ .

The differential equation needed (see p. 525) is

$$y' = -k(y - T_0)$$

which has general solution

$$y(t) = T_0 + Ce^{-kt}$$

so

$$y(t) = T_0 + Ce^{-kt} \quad (\text{temperature of water})$$

We know that  $\lim_{t \rightarrow \infty} y(t) = T_0$ , so  $T_0 = 60$ . Also,

$$y(20) = 60 + Ce^{-20k} = 100 \Rightarrow Ce^{-20k} = 40$$

$$y(60) = 60 + Ce^{-60k} = 80 \Rightarrow Ce^{-60k} = 20$$

$$\therefore \frac{Ce^{-20k}}{Ce^{-60k}} = \frac{40}{20} = 2$$

$$\therefore e^{-20k+60k} = 2$$

$$\therefore \ln(e^{40k}) = \ln(2)$$

$$40k = \ln(2)$$

$$\therefore k = \frac{\ln(2)}{40}$$

(b) What is the differential equation satisfied by the temperature of the bar?

$$y' = -\frac{\ln(2)}{40} (y - 60)$$

(c) What is a formula for  $F(t)$ ?

$$y = F(t) = 60 + Ce^{-\frac{\ln(2)}{40}t}$$

To find  $C$ , we have

$$F(20) = 60 + Ce^{-\frac{\ln(2)}{40} \cdot 20} = 100$$

$$\therefore Ce^{-\frac{\ln(2)}{2}} = 40$$

$$C(e^{\ln(2)})^{-\frac{1}{2}} = 40$$

$$C \cdot (2)^{-\frac{1}{2}} = 40$$

$$C = 40\sqrt{2}$$

$$\therefore F(t) = 60 + 40\sqrt{2} e^{-\frac{\ln(2)}{40}t}$$

(d) Determine the temperature of the bar at the moment it is submerged.

$$F(0) = 60 + 40\sqrt{2} e^{-\frac{\ln(2)}{40} \cdot 0}$$

$$= 60 + 40\sqrt{2} e^0 = \boxed{60 + 40\sqrt{2}} (\text{ }^\circ\text{F})$$

$$\approx 116.2 \text{ }^\circ\text{F}$$

9.2.11

A 60-kg skydiver jumps out of an airplane. What is her terminal velocity in mph, assuming that  $k = 10 \text{ kg/s}$  (no parachute)?

By equation (4) (p. 526), with  $k = 10 \text{ kg/s}$ ,  $g = 9.8 \text{ m/s}^2$ , and  $m = 60 \text{ kg}$ ,

$$\begin{aligned}\text{Terminal velocity} &= -\frac{gm}{k} \\ &= -\frac{9.8 \text{ m/s}^2 \cdot 60 \text{ kg}}{10 \text{ kg/s}} \\ &= \boxed{-58.8 \frac{\text{m}}{\text{s}}}\end{aligned}$$

9.2.17

Find the minimum initial deposit that will allow an annuity to pay out \$500 / year indefinitely if it earns interest at 5%.

On p. 527, we are told that the general solution for the balance  $P(t)$  in an annuity which earns interest at rate  $r$  and pays out continuously at  $N$  dollars per year is

$$P(t) = \frac{N}{r} + Ce^{rt}$$

The annuity goes to zero if  $C < 0$  and increases indefinitely if  $C > 0$ . Therefore the minimum initial deposit required for the annuity to pay out indefinitely will occur when  $C = 0$  so that

$$P(t) = \frac{N}{r}$$

The minimum initial deposit is then

$$P(0) = \frac{500}{.05} = \boxed{\$10,000}$$

\*Note: The amount of money in the account will actually remain constant at \$10,000.

9.2.22

Let  $N(t)$  be the fraction of the population who have heard a given piece of news  $t$  hours after its initial release. According to one model, the rate  $N'(t)$  at which the news spreads is equal to  $k$  times the fraction of the population that has not yet heard the news, for some constant  $k$ .

(a) Determine the differential equation satisfied by  $N(t)$ .

The fraction of the population that hasn't heard the news at time  $t$  is  $1 - N(t)$ . Therefore we have

$$N'(t) = k(1 - N(t)) = -k(N(t) - 1)$$

(b) Find the solution of this differential equation with the initial condition  $N(0) = 0$  in terms of  $k$ .

We are given in this section that the general solution for  $N(t)$  is

$$N(t) = 1 + Ce^{-kt}$$

so using the fact that  $N(0) = 0$  yields

$$N(0) = 0 = 1 + Ce^0 = 1 + C \Rightarrow C = -1$$

$$\therefore \boxed{N(t) = 1 - e^{-kt}}$$

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate  $k$  and estimate the percentage that will know

about the earthquake 12 hours after it occurs.

We are given that  $N(8) = \frac{1}{2}$ , so

$$N(8) = \frac{1}{2} = 1 - e^{-8k}$$

$$e^{-8k} = \frac{1}{2}$$

$$\ln(e^{-8k}) = \ln\left(\frac{1}{2}\right)$$

$$-8k = \ln 1 - \ln 2 = -\ln 2$$

$$\therefore k = \boxed{\frac{\ln 2}{8}}$$

The percentage (i.e. fraction of the population which will know about the earthquake 12 hours after it occurs is

$$N(12) = 1 - e^{-\frac{\ln 2}{8} \cdot 12}$$

$$= 1 - e^{(\ln 2) \cdot (-\frac{3}{2})}$$

$$= 1 - e^{\ln(2^{-\frac{3}{2}})}$$

$$= 1 - 2^{-\frac{3}{2}}$$

$$= \boxed{1 - \frac{1}{2\sqrt{2}}} \approx 64.63\%$$

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9.4.1

Find the general solution of the logistic equation

$$y' = 3y \left(1 - \frac{y}{5}\right)$$

Then find the particular solution satisfying  $y(0)=2$ .

Here we have  $k=3$  and  $A=5$ , so that

$$y = \frac{5}{1 - e^{-3t}/c}$$

is the general solution and  $y(0)=2$  implies

$$y(0) = 2 = \frac{5}{1 - e^0/c}$$

$$2 = \frac{5}{1 - 1/c}$$

$$2 - \frac{2}{c} = 5$$

$$2 - 5 = \frac{2}{c}$$

$$-3 = \frac{2}{c}$$

$$c = -\frac{2}{3}$$

$$\therefore y = \frac{5}{1 + \frac{3}{2}e^{-3t}}$$

9.4.6

The population  $P(t)$  of mosquito larvae growing in a tree hole increases according to the logistic equation with growth constant  $k=0.3 \text{ days}^{-1}$  and carrying capacity  $A=500$ .

(a) Find a formula for the larvae population  $P(t)$ , assuming an initial population of  $P_0=50$  larvae.

We have

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right) = 0.3y \left(1 - \frac{y}{500}\right)$$

where  $y = P(t)$ . Hence

$$P(t) = \frac{500}{1 - e^{-0.3t}/C}$$

so since  $P_0 = 50$ ,

$$P(0) = 50 = \frac{500}{1 - e^0/C}$$

$$50\left(1 - \frac{1}{C}\right) = 500$$

$$1 - \frac{1}{C} = 10$$

$$1 - 10 = \frac{1}{C}$$

$$-9 = \frac{1}{C} \Rightarrow C = -\frac{1}{9}$$

$$\boxed{P(t) = \frac{500}{1 + 9e^{-0.3t}}}$$

(b) After how many days will the larvae population reach 200?

We need to find the time  $t$  at which  $P(t) = 200$ , so

$$P(t) = 200 = \frac{500}{1 + 9e^{-0.3t}}$$

$$1 + 9e^{-0.3t} = \frac{5}{2}$$

$$9e^{-0.3t} = \frac{3}{2}$$

$$e^{-0.3t} = \frac{1}{6}$$

$$\ln(e^{-0.3t}) = \ln\left(\frac{1}{6}\right)$$

$$-0.3t = \ln 1 - \ln 6 = -\ln 6$$

$$\boxed{t = \frac{10 \ln 6}{3}} \approx 5.97 \text{ days}$$

9.4.9

A rumor spreads through a school with 1,000 students. At 8 AM, 80 students have heard the rumor and by noon, half the school has heard it. Using the logistic model of Exercise 8, determine when 90% of the students will have heard the rumor.

We will measure time  $t$  in hours and let 8 AM correspond to time zero. The carrying capacity of the school ( $A$ ) is 1,000. Therefore, if we let  $y(t)$  be the number of students who have heard the rumor at time  $t$ , we have

$$y(t) = \frac{1000}{1 - e^{-kt}/c}$$

We know that  $y(0) = 80$ , so

$$y(0) = 80 = \frac{1000}{1 - e^0/c}$$

$$\therefore 80 \left(1 - \frac{e^0}{c}\right) = 1000$$

$$\therefore 1 - \frac{1}{c} = \frac{25}{2}$$

$$1 - \frac{25}{2} = \frac{1}{c}$$

$$\frac{-23}{2} = \frac{1}{c}$$

$$\therefore c = \frac{-2}{23}$$

$$\therefore y(t) = \frac{1000}{1 + \frac{23}{2} e^{-kt}}$$

We finally must find  $k$ , so we use the fact that  $y(4) = 500$ :

$$y(4) = 500 = \frac{1000}{1 + \frac{23}{2} e^{-4k}}$$

$$1 + \frac{23}{2} e^{-4k} = 2$$

$$\frac{23}{2} e^{-4k} = 1$$

$$e^{-4k} = \frac{2}{23}$$

$$\ln(e^{-4k}) = \ln\left(\frac{2}{23}\right)$$

$$-4k = \ln\left(\frac{2}{23}\right)$$

$$k = -\frac{1}{4} \ln\left(\frac{2}{23}\right)$$

We thus substitute this value of  $k$  into our equation and find the value of  $t$  for which

$$y(t) = 900:$$

$$y(t) = 900 = \frac{1000}{1 + \frac{23}{2} e^{\frac{1}{4} \ln\left(\frac{2}{23}\right)t}}$$

$$1 + \frac{23}{2} e^{\frac{1}{4} \ln\left(\frac{2}{23}\right)t} = \frac{10}{9}$$

$$\frac{23}{2} e^{\frac{1}{4} \ln\left(\frac{2}{23}\right)t} = \frac{1}{9}$$

$$e^{\frac{1}{4} \ln\left(\frac{2}{23}\right)t} = \frac{2}{207}$$

$$\frac{1}{4} \ln\left(\frac{2}{23}\right)t = \ln\left(\frac{2}{207}\right)$$

$$\therefore t = 4 \cdot \frac{\ln\left(\frac{2}{207}\right)}{\ln\left(\frac{2}{23}\right)} \approx 7.6 \text{ hours (3:36 P.M.)}$$