- Homework #3
  - **Section 11.2:** 12, 18, 20
  - **Section 13.3:** 6, 10, 18, 23
  - **Section 13.5:** 5, 12, 14, 16, 18, 26, 27, 41
  - **Section 14.1:** 8, 16, 18, 20, 24, 28, 32, 44

$(1-\cos\theta)^2 + \sin^2\theta =$

$1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$

11.2

12) $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$c(\theta) = (5(\theta - \sin\theta), 5(1-\cos\theta))$

$0 \le \theta < 2\pi$

$c'(\theta) = (5(1-\cos\theta), 5(\sin\theta))$

$s = \int_0^{2\pi} \sqrt{5^2(1-\cos\theta)^2 + 5^2(\sin\theta)^2}\, d\theta$

$s = 5 \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$= 5\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$= 5 \int_0^{2\pi} (2 - 2\cos\theta)^{1/2} d\theta$

~~$10\pi 5^2$~~

$\int (a + f(x))^{1/2} dx \qquad a + f(x) = g(t)$

$f(\theta) = 2 - 2\cos\theta$

$g(\theta) = \theta^{1/2}$

$g(f(\theta)) = (2 - 2\cos\theta)^{1/2}$

$[f(g(x))]' = g'(x) f'(g(x))$

18) $(5t + 1, 4t - 3) \quad t = 9$

$\frac{ds}{dt} = \sqrt{5^2 + 4^2}$

$= \sqrt{25 + 16}$

$= \sqrt{41}$

$= 5 \int_0^{2\pi} \sqrt{1 + (1 - 2\cos\theta)}\, d\theta$

$= 5 \int_0^{2\pi} \sqrt{1 + 2\left(\frac{1 - 2\cos\theta}{2}\right)}$

$1 - \cos\theta = 1 - \cos 2(\tfrac{1}{2}\theta)$

$= 2\left(\frac{1 - \cos 2(\frac{1}{2}\theta)}{2}\right)$

$= 2\sin^2(\tfrac{1}{2}\theta)$

$= 5 \int_0^{2\pi} (4\sin^2(\tfrac{1}{2}\theta))^{1/2} d\theta$

$= 10 \int_0^{2\pi} \sin(\tfrac{1}{2}\theta)\, d\theta$

$= 10 [2(-\cos(\tfrac{1}{2}\theta)]_0^{2\pi}$

$= 20 [-\cos(\pi) + \cos(0)]$

$= 20 [1 + 1]$

$= 40$

1.2

18) $(5t+1, 4t-3)$ $\quad t=9$

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$$

$$= \sqrt{5^2 + 4^2}$$

$$= \sqrt{25+16}$$

$$= \sqrt{41} \quad \text{for any time } t$$

in particular $\frac{ds}{dt} = \sqrt{40}$ at $t=9$.

20)

$$c(t) = (r\cos\omega t, r\sin\omega t)$$

$$\frac{ds}{dt} = \sqrt{(-r\omega\sin\omega t)^2 + (r\omega\cos\omega t)^2}$$

$$= \sqrt{r^2\omega^2\sin^2\omega t + r^2\omega^2\cos^2\omega t}$$

$r\omega$

13.3

6) $r(t) = \langle t\cos t, t\sin t, 3t\rangle \quad 0 \le t \le 2\pi$

$$L = \int_a^b \|r'(t)\|\,dt$$

$$\cancel{= \int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t + 9t^2}\,dt}$$

$$\cancel{= \int_0^{2\pi} t\sqrt{1+9}\,dt}$$

$$\cancel{= \sqrt{10}\int_0^{2\pi} t\,dt = \sqrt{10}\left[\frac{t^2}{2}\right]_0^{2\pi} = \sqrt{10}\,\frac{4\pi^2}{2}}$$

$$= (\sqrt{10})2\pi^2$$

~~10) $r(t) = \langle\sin 3t, \cos 4t, \cos 5t\rangle$~~

~~$\frac{ds}{dt} = \|r'(t)\|$~~

~~$s(t) = \sqrt{\sin}$~~

$$r'(t) = \langle t(-\sin t) + \cos t,\ t(\cos t) + \sin t,\ 3\rangle$$

$$L = \int_0^{2\pi} \sqrt{(t(-\sin t) + \cos t)^2 + (t\cos t + \sin t)^2 + 9}\,dt$$

$$= \int_0^{2\pi} \sqrt{t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t + t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t + 9}\,dt$$

$$= \int_0^{2\pi} \sqrt{t^2+10}\,dt \qquad \cancel{u = t^2+10}$$

By Formula # 84

$$= \left[\frac{\sqrt{10}}{2}\sqrt{t^2+10} + \frac{10}{2}\ln\left(t + \sqrt{t^2+10}\right)\right]_0^{2\pi}$$

10)

$$r(t) = \langle\sin 3t, \cos 4t, \cos 5t\rangle \qquad t = \frac{\pi}{2}$$

$$r'(t) = \langle 3\cos 3t, 4(-\sin 4t), 5(-\sin 5t)\rangle$$

$$v(t) = \|r'(t)\|$$

$$= \sqrt{9\cos^2 3t + 16\sin^2 4t + 25\sin^2 5t}$$

$$v\left(\frac{\pi}{2}\right) = \sqrt{9\cos^2\left(\frac{3\pi}{2}\right) + 16\sin^2(2\pi) + 25\sin^2\left(\frac{5\pi}{2}\right)}$$

$$= \sqrt{9(0) + 16(0) + 25(1)^2}$$

$$= 5$$

$\boxed{13.3}$

18 (see pg 626)

$r(t) = \langle t - \sin t, 1 - \cos t \rangle$

$L = \int_a^b \|r'(t)\|$

$r'(t) = \langle 1 - \cos t, \sin t \rangle$

One arch runs from $t = 0$ to $t = 2\pi$.

Verify this with the def on pg 626

$$L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{4(\sin^2(\tfrac{1}{2}t))}\, dt$$

$$= 2\int_0^{2\pi} \sin \tfrac{1}{2}t\, dt$$

$$= 2\left[2(-\cos \tfrac{1}{2}t)\right]_0^{2\pi}$$

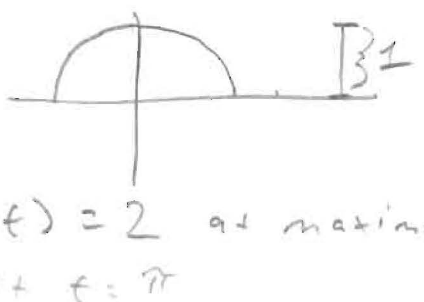
$$= 4[-\cos \pi + \cos 0]$$

$$= 4[-(-1) + 1]$$

$$= 8 \checkmark$$

$v(t) = \|r'(t)\| = 2\sin(\tfrac{1}{2}t)$

$\sin(\tfrac{1}{2}t)$ runs from 0 to 1 on $0 \le t \le 2\pi$



So $v(t) = 2$ at maximum at $t = \pi$

23

a) The one w/ more turns.

b)

Spring one:

$c_1(t) = \left\langle 5\cos\left(\frac{2\pi(3)t}{4}\right), 5\sin\left(\frac{2\pi(3)t}{4}\right), t \right\rangle$
$0 \le t \le 4$

$L = \int_a^b \|r'(t)\|$

~~Spring~~

$c_1'(t) = \left\langle -5\sin\left(\frac{6\pi t}{4}\right)\cdot\left(\frac{3\pi}{2}\right), 5\cos\left(\frac{3\pi t}{2}\right)\cdot\left(\frac{3\pi}{2}\right), 1 \right\rangle$

$$L_1 = \int_0^4 \sqrt{25\left(\frac{9\pi^2}{4}\right)\sin^2\left(\frac{3\pi t}{2}\right) + 25\left(\frac{9\pi^2}{4}\right)\cos^2\left(\frac{3\pi t}{2}\right) + 1}\, dt$$

$$= \int_0^4 \sqrt{25\left(\frac{9\pi^2}{4}\right) + 1}\, dt$$

$$= 4\sqrt{25\left(\frac{9\pi^2}{4}\right) + 1} \approx 94.3$$

Spring two:

$$L_2 = h\sqrt{R^2\left(\frac{2\pi N}{h}\right)^2 + 1}$$

$$= 3\sqrt{4^2\left(\frac{2\pi(5)}{3}\right)^2 + 1}$$

$$= 3\sqrt{4^2\left(\frac{2^2 \cdot 5^2 \pi^2}{3^2}\right) + 1} = 125.7$$

which is bigger?

$\boxed{13.5}$

5 $r(\theta) = \langle \sin\theta, \cos\theta, \cos 3\theta \rangle$, $\theta = \frac{\pi}{3}$

$\vec{v}(\theta) = r'(\theta) = \langle \cos\theta, -\sin\theta, -3\sin 3\theta \rangle$

$\vec{v}(\frac{\pi}{3}) = r'(\frac{\pi}{3}) = \langle \cos\frac{\pi}{3}, -\sin\frac{\pi}{3}, -3\sin\pi \rangle$

$= \langle \frac{1}{2}, -\frac{\sqrt{3}}{2}, -3\cancel{\left(\frac{\sqrt{3}}{2}\right)}(0) \rangle$

$v(\frac{\pi}{3}) = \|\vec{v}(\frac{\pi}{3})\| = \sqrt{\frac{1}{4} + \frac{3}{4} + \cancel{9\cdot\frac{3}{4}}} = \cancel{\frac{\sqrt{31}}{2}}\ 1$

$\vec{a}(\theta) = r''(\theta) = \langle -\sin\theta, -\cos\theta, -9\cos 3\theta \rangle$

$\vec{a}(\frac{\pi}{3}) = \ldots$

$\boxed{13.5}$

12)

$\vec{a}(t) = \langle e^t, 0, t+1 \rangle$, $\vec{v}(0) = \langle 1, -3, \sqrt{2} \rangle$

$\vec{v}(t) = \langle \int e^t dt, \int 0 dt, \int (t+1) dt \rangle$

$= \langle e^t + c_1, 0 + c_2, \frac{t^2}{2} + t + c_3 \rangle$

$\vec{v}(0) = \langle 1 + c_1, c_2, c_3 \rangle = \langle 1, -3, \sqrt{2} \rangle$

$\Rightarrow 1 + c_1 = 1 \Rightarrow c_1 = 0$

$c_2 = -3$

$c_3 = \sqrt{2}$

$\vec{v}(t) = \langle e^t, -3, \frac{t^2}{2} + t + \sqrt{2} \rangle$

14) $\vec{a}(t) = \langle 0, 0, t^2 \rangle$

$\vec{v}(0) = \langle 1, -1, 0 \rangle$

$\vec{v}(t) = \langle c_1, c_2, \frac{t^3}{3} + c_3 \rangle$

$c_1 = 1$

$c_2 = -1$

$c_3 = 0$

16) $\vec{a}(t) = \langle e^t, 2t, t+1 \rangle$

$\vec{v}(0) = \langle 1, 0, 1 \rangle$

$\vec{r}(0) = \langle 2, 1, 1 \rangle$

$\vec{v}(t) = \langle e^t + c_1, t^2 + c_2, \frac{t^2}{2} + t + c_3 \rangle$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 1$

$\vec{r}(t) = \langle e^t + d_1, \frac{t^3}{3} + d_2, \frac{t^3}{6} + \frac{t^2}{2} + t + d_3 \rangle$

$d_1 = 1 \quad d_2 = 1 \quad d_3 = 1$

18)

$\vec{a}(t) = \langle 0, 0, \cos t \rangle$

$\vec{v}(0) = \langle 1, -1, 0 \rangle$

$\vec{r}(0) = \langle 1, 0, 0 \rangle$

23)

$80 = \left(\frac{60^2}{g}\right) \sin 2\theta$

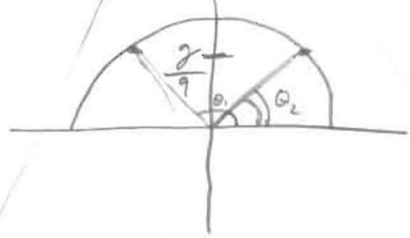
$2^4 \cdot 5 = \frac{2^4 3^2 5^2}{g} \sin 2\theta$

$\frac{g}{9} = \sin 2\theta$

$\sin^{-1}\left(\frac{g}{9}\right) = 2\theta$

$\frac{\sin^{-1}\left(\frac{g}{9}\right)}{2} = \theta$

There are 2 such θ's b/t $0, \frac{\pi}{2}$.



26) $\vec{v} = \langle \overset{2^2 \cdot 3}{12}, \overset{2^2 5}{20}, \overset{2^2 5}{20} \rangle$ $\vec{a} = \langle 2, 1, -3 \rangle$

$\|\vec{v}\| = \sqrt{12^2 + 20^2 + 20^2}$

$= \sqrt{2^4 \cdot 3^2 + 2^4 \cdot 5^2 + 2^4 \cdot 5^2}$

$= 2^2 \sqrt{9 + 50}$

$= 4\sqrt{59} \quad (1)$

$\vec{v} + \vec{a} = \langle 14, 21, 17 \rangle$

$\|\vec{v} + \vec{a}\| = \sqrt{14^2 + 21^2 + 17^2} \quad (2)$

which is bigger? (1) or (2)?

OR...

$(\|\vec{v}\|^2)' = (v \cdot v)' = 2v' \cdot v = 2 \cdot a \cdot v$

$= \cdots = -32 < 0$

$\boxed{13.5}$

27) a) Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\bar{v} = \frac{1}{T}\int_0^T \vec{r}'(t)\,dt = 0$

$\Rightarrow \int_0^T \vec{r}'(t)\,dt = 0$

$\Rightarrow \int_0^T \langle x'(t), y'(t), z'(t) \rangle dt = 0$

$\Rightarrow \left\langle \int_0^T x'(t)dt, \int_0^T y'(t)dt, \int_0^T z'(t) \right\rangle = 0$

$\Rightarrow$ By the Fund. Thm of Calc:

$\langle x(T)-x(0), y(T)-y(0), z(T)-z(0) \rangle = 0$

$\Rightarrow x(T) = x(0)$
$y(T) = y(0)$
$z(T) = z(0)$

So the particle is again @ the origin.

b) No.

Ex:

If $\vec{r}(t) = \langle \sin t, 0, 0 \rangle$, $T = 2\pi$

$\vec{r}'(t) = \langle \cos t, 0, 0 \rangle$

$\bar{v} = \frac{1}{2\pi}\int_0^{2\pi} \langle \cos t, 0, 0 \rangle dt$

$= \frac{1}{2\pi}\left\langle \int_0^{2\pi} \cos t\,dt, 0, 0 \right\rangle$

$= \frac{1}{2\pi}\langle \sin 2\pi - \sin 0, 0, 0 \rangle$

$= \frac{1}{2\pi}\langle 0, 0, 0 \rangle = 0$

But! the speed is always positive (non-negative)

$v(t) = \|\vec{r}'(t)\| = \sqrt{\cos^2 t + 0 + 0}$
$= |\cos t|$

and e.g. $v(0) = 1$.

So the average speed is (strictly) positive.

41) $\vec{r}(t) = \langle t, \cos t, t\sin t \rangle$, $t = \frac{\pi}{2}$

Step 1: Compute $\vec{T}$ & a_T

$\vec{v}(t) = \vec{r}'(t) = \langle 1, -\sin t, t\cos t + \sin t \rangle$

$\vec{a}(t) = \vec{r}''(t) = \langle 0, -\cos t, t(-\sin t) + \cos t + \cos t \rangle$

At $t = \frac{\pi}{2}$

$\vec{v} = \vec{r}'(\frac{\pi}{2}) = \langle 1, -1, \frac{\pi}{2}(0) + 1 \rangle$
$= \langle 1, -1, 1 \rangle$

$\vec{a} = \vec{r}''(\frac{\pi}{2}) = \langle 0, 0, -\frac{\pi}{2} \rangle$

$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{3}}$

$a_T = \vec{a}\cdot\vec{T} = 0 + 0 + (-\frac{\pi}{2})(\frac{1}{\sqrt{3}})$
$= -\frac{\pi}{2\sqrt{3}}$

Step 2: Compute a_N & N.

$a_N N = a - a_T T$
$= \langle 0, 0, -\frac{\pi}{2} \rangle - \pi\langle -\frac{1}{6}, \frac{1}{6}, \frac{-1}{6} \rangle$
$= \langle \frac{1}{6}, \cancel{\frac{-1}{6}}, \frac{-\pi}{2} + \frac{1}{6} \rangle \ldots = \frac{\pi}{6}(1, -1, -2)$

$a_N = \|a_N N\| = \ldots = \frac{\pi}{\sqrt{6}}$
$= \sqrt{\frac{1}{36} + \cancel{\frac{1}{36}} + \frac{(-3\pi+1)^2}{36}}$

$N = \frac{a_N N}{a_N} = \cancel{\langle \frac{1}{6}, \frac{-1}{6}, \frac{-\pi}{2} + \frac{1}{6} \rangle / a_N}$

$\ldots = \frac{1}{\sqrt{6}}(1, -1, -2)$

Step 3: Write the decomp:

$\vec{a} = \langle 0, 0, -\frac{\pi}{2} \rangle = a_T T + a_N N$