14.4	14.5	14.6
4	2	2
6	4	4
10	22	10
14	24	12
18	28	16
20	30	28
22	38	30
	40	35
	42	
	50	
	60	

$\boxed{14.4}$ (Note: "..." means "continue in this way".)

4) $f(3,2)=2$, $f_x(3,2)=-1$, $f_y(3,2)=3$

$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

$f(3.1,1.8)\approx f(3,2)+f_x(3,2)(3.1-3)+f_y(3,2)(1.8-2)$

$= 2+(-1)(.1)+3(-.2)$

$= 2-.1-.6 = \boxed{1.3}$

6) $f(x,y)=\dfrac{x^2}{y^2+1}$ $\quad f_x(x,y)=\dfrac{2x}{y^2+1}$ $\quad f_y(x,y)=\dfrac{-x^2(2y)}{(y^2+1)^2}$

$f(4,1)=\dfrac{16}{2}=8$ $\quad f_x(4,1)=\dfrac{8}{2}=4$ $\quad f_y(4,1)=\dfrac{-16(2)}{4}=-8$

$f(4.01,0.98)\approx 8+4(4.01-4)+(-8)(0.98-1)$

...

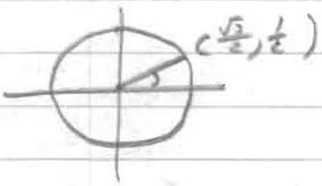
$\boxed{14.4}$

10) $f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b)$
$+ f_z(a,b,c)(z-c)$

. . .

14) $G(u,w) = \sin(uw), \quad (\frac{\pi}{6}, 1)$

$G_u(u,w) = w\cos(uw)$



$G_w(u,w) = u\cos(uw)$

$G_u(\frac{\pi}{6}, 1) = 1\cos(\frac{\pi}{6}\cdot 1) = \frac{\sqrt{3}}{2}$

$G_w(\frac{\pi}{6}, 1) = \frac{\pi}{6}\cdot\cos(1\cdot\frac{\pi}{6}) = \frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}$

$G(\frac{\pi}{6}, 1) = \sin(\frac{\pi}{6}\cdot 1) = \frac{1}{2}$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$z = G(\frac{\pi}{6}, 1) + G_u(\frac{\pi}{6}, 1)(u - \frac{\pi}{6}) + G_w(\frac{\pi}{6}, 1)(w-1)$

. . .

18) $f(x,y) = \ln(x^2+y^2), \quad (1,1)$
$f(1,1) = \ln(2)$
$f_x(x,y) = \frac{1}{x^2+y^2}(2x), \quad f_x(1,1) = \frac{2}{2} = 1$

$f_y(x,y) = \frac{1}{x^2+y^2}(2y), \quad f_y(1,1) = \frac{2}{2} = 1$

. . .

20) $z = x^2 e^y, \quad 5x - 2y + \frac{1}{2}z = 0$
Let $f(x,y) = x^2 e^y$.
The equation for the plane tangent to $z = f(x,y)$ at the point (a,b) is:
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

14.4

20) cont...:

$$f_x(x,y) = 2xe^y \quad , \quad f_y(x,y) = x^2 e^y$$

So the tangent plane at (a,b) is:

$$z = a^2e^b + (2ae^b)(x-a) + (a^2e^b)(y-b)$$

Let (a,b) be a point where the tangent plane is parallel to $5x - 2y + \frac{1}{2}z = 0$, then

① $c = a^2e^b$

② $z = a^2e^b + (2ae^b)(x-a) + (a^2e^b)(y-b)$

③ $(5,-2,\frac{1}{2})(x,y,z) = (5,-2,\frac{1}{2})(a,b,c)$

① is what we get when we plug a & b into $f(x,y)$.
② is the equation for a tangent plane at (a,b).
③ is the equation for a plane parallel to $5x-2y+\frac{1}{2}z=0$ that passes through (a,b).

Plugging ① into ②, we get:

$$z = c + 2\frac{c}{a}(x-a) + c(y-b)$$

This isn't working; let's try something different!

In order to be parallel to $5x-2y+\frac{1}{2}z=0$, the tangent plane would have to have a normal vector that is some (nonzero) multiple of $\vec{n} = (5,-2,\frac{1}{2})$. That is, the tangent ~~vector~~ plane has normal vector $\lambda\vec{n} = (5\lambda, -2\lambda, \frac{1}{2}\lambda)$, $\lambda \neq 0$.

14.4 26 cont...:

But the tangent plane has equation

$z = a^2 e^b + (2ae^b)(x-a) + (a^2e^b)(y-b)$

Or, rearranging:

$-a^2e^b = (2ae^b)(x-a) + a^2e^b(y-b) - z$

So its normal vector is:

$\vec{m} = (2ae^b, a^2e^b, -1)$

Setting $\lambda\vec{n} = \vec{m}$ we get:

$2ae^b = 5\lambda$
$a^2e^b = -2\lambda$
$-1 = \frac{1}{2}\lambda \Rightarrow \lambda = \frac{-1}{2}$

So $2ae^b = \frac{-5}{2} \Rightarrow ae^b = \frac{-5}{4}$
$a^2e^b = +1$

So $+1 = a^2e^b = a(ae^b) = a(\frac{-5}{4}) \Rightarrow \boxed{a = \frac{-4}{5}}$

So $\frac{-5}{2} = 2ae^b = 2(-\frac{4}{5})e^b$
$\Rightarrow \frac{+5^2}{2^4} = e^b \Rightarrow b = \ln\left(\frac{+5^2}{2^4}\right)$
$= \boxed{2\ln 5 - 4\ln 2}$

So the (only) point where this is true is:

$\left(-\frac{4}{5}, \ln\left(\frac{5^2}{2^4}\right), \left(\frac{-4}{5}\right)^2 \cdot e^{\left(\ln\left(\frac{5^2}{2^4}\right)\right)}\right)$
$= \left(-\frac{4}{5}, \ln\left(\frac{5^2}{2^4}\right), \left(\frac{16}{25}\right)\cdot\left(\frac{25}{16}\right)\right) = \left(-\frac{4}{5}, \ln\left(\frac{5^2}{2^4}\right), 1\right)$

$\boxed{14.4}$

22) Let $z = f(x,y)$ be a surface.
At the point $(-2,3,4)$, the tangent plane is $z + 4x + 2y = 2$. So the height of the surface is approximated by

$$z = -4x - 2y + 2$$

That is $f(x,y) \approx -4x - 2y + 2$ near the point $(-2,3,4)$.

(Let's check that $(-2,3,4)$ is actually on the given plane: That is, does $4 = -4(-2) - 2(3) + 2$?
$8 - 6 + 2 = 4$ ✓)

So $f(-2.1, 3.1) \approx -4(-2.1) - 2(3.1) + 2$
$= \cdots$

$\boxed{14.5}$

2) $f(x,y) = e^{xy}$ $\quad c(t) = (t^3, 1+t)$

a) $\nabla f = \langle f_x, f_y \rangle = \langle y e^{xy}, x e^{xy} \rangle$

$\vec{c}\,'(t) = \langle 3t^2, 1 \rangle$

b) $\frac{d}{dt} f(c(t)) = \nabla f_{c(t)} \cdot c'(t) = (*)$

$\nabla f_{c(t)} = \langle y e^{xy}, x e^{xy} \rangle \big|_{(t^3, 1+t)}$
$= \langle (1+t) e^{t^3(1+t)}, t^3 e^{t^3(1+t)} \rangle$

$(*) = \langle (1+t) e^{t^3(1+t)}, t^3 e^{t^3(1+t)} \rangle \cdot \langle 3t^2, 1 \rangle$

$\boxed{14.5}$

2) b) cont...

$$= (1+t)e^{t^3(1+t)}(3t^2) + t^3 e^{t^3(1+t)}$$

c) $f(c(t)) = \exp(t^3(1+t))$

$$\frac{d}{dt}(f(c(t))) = \exp(t^3(1+t))(3t^2(1+t) + t^3(1))$$

4) $A = 0$, $B = \text{neg}$, $C = \text{pos}$, $D = 0$.

22) $f(x,y) = x^2y^3$, $\vec{v} = (1,1)$, $P = (-2,1)$

$$D_u f(P) = \frac{1}{\|v\|} D_v f(P)$$

$$= \frac{1}{\|v\|}(\nabla f_p \cdot v)$$

$$= \frac{1}{\sqrt{2}}(\langle 2xy^3, x^2(3y^2)\rangle|_p \cdot \langle 1,1\rangle)$$

$$= \frac{1}{\sqrt{2}}(2xy^3 + x^2(3y^2))|_p$$

$$= \frac{1}{\sqrt{2}}(2(-2)(1)^3 + (-2)^2(3(1)^2))$$

---

24) $f(x,y) = \sin(x-y)$, $v = \langle 1,1\rangle$, $P = (\frac{\pi}{2}, \frac{\pi}{6})$

$$D_u f(P) = \frac{1}{\|v\|} D_v f(P)$$

$$= \frac{1}{\|v\|}(\nabla f_p \cdot v)$$

$$= \frac{1}{\sqrt{2}}(\langle \cos(x-y), -\cos(x-y)\rangle|_p \cdot \langle 1,1\rangle)$$

$$= \frac{1}{\sqrt{2}}(\cos(x-y) - \cos(x-y))|_p = 0$$

28) $g(x,y,z) = z^2 - xy^2$, $v = \langle -1, 2, 2 \rangle$, $P = (2,1,3)$

. . .

30) $g(x,y,z) = x \ln(y+z)$, $v = \langle 2, -1, 1 \rangle$

$P = (2, e, e)$ Note: e is the constant such that $\ln(e) = 1$.

. . .

38) $\nabla f_p = \langle 2x, 2y, -2z \rangle |_p = \langle 6, 2, -4 \rangle$

40) $\nabla f_p = \langle \frac{x}{2}, \frac{2y}{9}, 2z \rangle$ is a vector normal to the ellipsoid at the point (x, y, z).

If Δf is a non-zero scalar multiple of $v = \langle 1, 1, -2 \rangle$ then we have our points.

That is, let's solve $\Delta f = \lambda \langle 1, 1, -2 \rangle$

So $\frac{x}{2} = \lambda$, $\frac{2y}{9} = \lambda$, $2z = -2\lambda$

$x = 2\lambda$, $y = \frac{9}{2}\lambda$, $z = -\lambda$

But we also have to have:

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

$$\Rightarrow \frac{\cancel{4}\lambda^2}{\cancel{4}} + \frac{\frac{9^{\cancel{2}}}{2^2}\lambda^2}{\cancel{9}} + \lambda^2 = 1$$

$$\frac{17}{4}\lambda^2 = 1$$

$$\lambda = \pm\sqrt{\frac{4}{17}}$$

check the calculations! ✓

So $\pm\sqrt{\frac{4}{17}} \langle 2, \frac{9}{2}, -1 \rangle$ are the 2 points.

$\boxed{145}$

42) $xz + 2x^2y + y^2z^3 = 11$, $P = (2,1,1)$

Let $f = xz + 2x^2y + y^2z^3$

$$\Delta f_p = \langle z + 4xy, 2x^2 + 2yz^3, x + 3y^2z^2 \rangle|_p$$
$$= \langle 1 + 4\cdot 2\cdot 1, 2\cdot 4 + 2\cdot 1\cdot 1^3, 2 + 3\cdot 1^2\cdot 1^2 \rangle$$
$$= \langle 9, 10, 5 \rangle$$

is the vector normal to the surface @ P.

So let $\vec{n} = \langle 9, 10, 5 \rangle$, then the tangent plane is given by:

$$\vec{n}\cdot\langle x,y,z\rangle = \vec{n}\cdot\overrightarrow{OP}.$$

50) $\nabla f = \langle z, 2y, x \rangle$

$\Rightarrow f = \langle zx + f_1(y,z), y^2 + f_2(x,z), xz + f_3(x,y) \rangle$

(for example: $f_1(y,z) = f_2(x,z) = f_3(x,y) = 0$)

60) Find the normal vectors of the two surfaces at the point, cross them to find a normal vector to both, and give the equation for a line in that direction through the point.

$$\vec{u} = \nabla f_p = \langle 3x^2 + 2y, 2x + z, y \rangle|_p = \langle 3+4, 2+1, 2 \rangle$$
$$= \langle 7, 3, 2 \rangle$$

$$\vec{v} = \nabla g_p = \langle 6x, -z, -y \rangle|_p$$
$$= \langle 6, -1, -2 \rangle$$

$\vec{u}\times\vec{v} = \cdots$

$\ell(t) = \overrightarrow{OP} + t(\vec{u}\times\vec{v}) \quad \cdots$

$\boxed{14.6}$

2) $f(x,y) = x\cos(y)$, $x = u^2+v^2$, $y = u-v$

a) $\frac{\partial f}{\partial x} = \cos(y)$

$\frac{\partial f}{\partial y} = -x\sin(y)$

b) $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

$= \cos(y)(2v) + (-x\sin(y))(-1)$

c) If $(u,v) = (2,1)$, then $(x,y) = (u^2+v^2, u-v)$
$= (2^2+1^2, 2-1)$
$= (5, 1)$

So $\frac{\partial f}{\partial v}\Big|_{(u,v)=(2,1)} = \cos(1)(2\cdot 1) + (-5\sin(1))(-1)$

$= 2\cos(1) + 5\sin(1)$

4) - - -

10) - - -

12) . . .

16) . . .

28) $\frac{\partial w}{\partial y}$, $\frac{1}{w^2+x^2} + \frac{1}{w^2+y^2} = 1$ @ $(x,y,w) = (1,1,1)$

$\boxed{14.6}$

28) cont...:

$$F(x,y,w) = \frac{1}{w^2+x^2} + \frac{1}{w^2+y^2}$$

$$F_y = \frac{-1}{(w^2+y^2)^2}\cdot(2y)$$

$$F_w = \frac{-1}{(w^2+x^2)^2}\cdot(2w) - \frac{1}{(w^2+y^2)^2}(2w)$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{-2y/(w^2+y^2)^2}{-2w/(w^2+x^2)^2-(2w)/(w^2+y^2)^2} \ldots$$

30) $\left(P+\frac{an^2}{V^2}\right)(V-nb) = nRT$ a, b, R constant

$\frac{\partial P}{\partial T}$; $\frac{\partial V}{\partial P}$

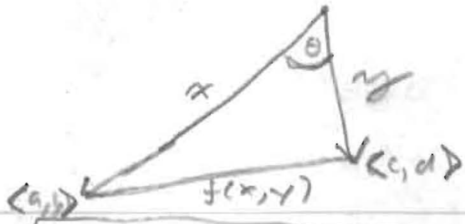
Let $F(P,V,T) = \left(P+\frac{an^2}{V^2}\right)(V-nb) - nRT$

$$F_T = -nR$$

$$F_P = V-nb$$

$$F_V = \left(\frac{an^2}{V^2}\cdot V - \frac{an^2nb}{V^2}\right)' = -\frac{an^2}{V^2} + \frac{an^3b}{V^3}(2V)$$

$$\frac{\partial P}{\partial T} = -\frac{F_T}{F_P} \qquad \frac{\partial V}{\partial P} = -\frac{F_P}{F_V} \quad \ldots$$



35 a) $f(x,y) = \sqrt{(a-c)^2 + (b-d)^2} = (*)$

$\langle a,b\rangle \cdot \langle c,d\rangle = \|\langle a,b\rangle\| \, \|\langle c,d\rangle\| \cos\theta$
$= x \, y \cos\theta$

$(*) = \sqrt{a^2 - 2ac + c^2 + b^2 - 2bd + d^2}$
$= \sqrt{(a^2+b^2) + (c^2+d^2) - 2(ac+bd)}$
$= \sqrt{x^2 + y^2 - 2\langle a,b\rangle \cdot \langle c,d\rangle}$
$= \sqrt{x^2 + y^2 - 2xy\cos\theta}$

b) $f(x,y) = \sqrt{x^2 + y^2 - xy}$

$\frac{\partial x}{\partial t} = V_a \qquad \frac{\partial y}{\partial t} = V_b$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$= \left[\frac{1}{2}(x^2+y^2-xy)^{-1/2}(2x-y)\right] V_a$

$+ \left[\frac{1}{2}(x^2+y^2-xy)^{1/2}(2y-x)\right] V_b$

Plug in $x = 30, \ y = 20, \ V_a = 4, \ V_b = 3$