Section 12.7

12) In cylindrical

$$r^2 \sin^2\theta + z^2 \le 4 \quad \text{and} \quad r\cos\theta = 0.$$

So $\theta = \pi/2, 3\pi/2$ or $r=0$, but if $r=0$ we may assume $\theta = \pi/2$ or $3\pi/2$.

Thus the equation is $r^2+z^2 \le 4$ and $\theta = \pi/2, 3\pi/2$

14) $x=y$ and $x, y \ge 0$ implies $\theta = \pi/4$

$x^2+y^2=9$ implies $r=3$ so cylindrically, we have $(3, \pi/4, z)$ where z is unconstrained

24) $z = r\cos\theta + r\sin\theta$ and so $r = \dfrac{z}{\cos\theta + \sin\theta}$

26) $\dfrac{(r\cos\theta)^2}{r\sin\theta\, z} = 1$. Simplifying $\dfrac{r\cos^2\theta}{\sin\theta\, z} = 1$ and so $r = \dfrac{\sin\theta\, z}{\cos^2\theta}$

42) $\rho^2 \le 1$

44) ~~$\rho = 1$~~ This is the surface of the unit sphere in the first octant so $\rho = 1,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/2$

56) $$(\rho\cos\phi)^2 = 3\left((\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2\right)$$

Simplifying

$$\rho^2\cos^2\phi = 3\rho^2\sin^2\phi, \text{ thus } \tan^2\phi = \frac{1}{3}$$

and $\phi = \pi/6, 5\pi/6$.

58) $$\rho\cos\phi = (\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2$$

Which simplifies to

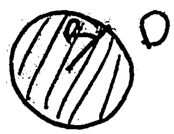
$$\rho\cos\phi = \rho^2\sin^2\phi \quad \text{and thus} \quad \rho = \frac{\cos\phi}{\sin^2\phi} = \cot\phi\csc\phi$$

15.4

2) 

$$\iint_D f\, dA = \int_0^{2\pi}\int_1^4 r^2\, r\, dr\, d\theta = \int_0^{2\pi} 64 - \tfrac{1}{4}\, d\theta$$

$$= \frac{255\pi}{2}$$

6) 

$$\iint_D f\, dA = \int_0^{2\pi}\int_0^R e^{r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(e^{R^2}-1)\, d\theta$$

$$= \pi(e^{R^2}-1)$$

8) $\boxed{a}$ The intersection of the two surfaces occurs when

$$x^2+y^2+(4-x^2-y^2)^2 = 6$$

Letting $r = x^2+y^2$,

$$r + (4-r)^2 = 6 \quad \text{or} \quad r^2 - 7r + 10 = 0$$

$$(r-5)(r-2) = 0$$

Thus there are two intersections, but for $2 < r < 5$ (say 3) the ~~circle~~ sphere is above the paraboloid, so the ~~intersection i~~ region is when $x^2+y^2 \le 2$.

$\boxed{b}$ The volume is $\int_0^{\sqrt{2}}\int_0^{2\pi} (\sqrt{6-r^2} + 4 - r)\, d\theta\, dr$

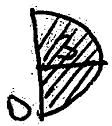
$$= 2\pi\int_0^{\sqrt{2}} 4r - r^2 - r\sqrt{6-r^2}\, dr$$

$$= 2\pi\left[2r^2 - \tfrac{1}{3}r^3 + \tfrac{1}{2}\cdot\tfrac{2}{3}(6-r^2)^{3/2}\right]_{r=0}^{\sqrt{2}}$$

$$= 2\pi\left[4 - \tfrac{2}{3}\sqrt{2} + \tfrac{1}{3}4^{3/2} - 0 - 0 - \tfrac{1}{3}6^{3/2}\right]$$

$$= 2\pi\left[\tfrac{20}{3} - \tfrac{2\sqrt{2}}{3} - \tfrac{1}{3}6^{3/2}\right]$$

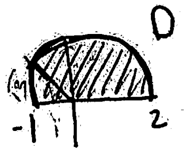
14)



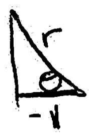
$$\int_0^3\int_0^{\sqrt{9-y^2}} \sqrt{x^2+y^2}\,dx\,dy = \int_0^3\int_{-\pi/2}^{\pi/2} r\,r\,d\theta\,dr$$

$$= \pi\int_0^3 r^2\,dr = 9\pi$$

16)



$$\int_{-1}^{2}\int_0^{\sqrt{4-x^2}} x^2+y^2\,dy\,dx = \int_0^2\int_0^{2\pi/3} r^2\,r\,d\theta\,dr + \int_{2\pi/3}^{\pi}\int_0^{-\sec\theta} r^2\,r\,dr$$

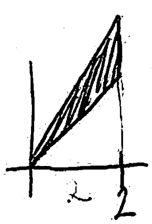


$r=-\sec\theta$

$$= \frac{2\pi}{3}\int_0^2 r^3\,dr + \int_{2\pi/3}^{\pi} \frac14 \sec^4\theta\,d\theta$$

$$= \frac{8\pi}{3} + \frac14\left[\frac13\tan\theta\sec^2\theta + \frac23\tan\theta\right]_{\theta=2\pi/3}^{\pi} = \frac{8\pi}{3} + \frac{\sqrt3}{2}$$

20)



$$\int_{\pi/4}^{\pi/3}\int_0^{2\sec\theta} \cancel{2\sec}\; r\cos\theta\; r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac13 (2\sec\theta)^3 \cos\theta\,d\theta$$

$$= \frac83\int_{\pi/4}^{\pi/3} \sec^2\theta\,d\theta = \frac83\left[\tan\theta\right]_{\pi/4}^{\pi/3} = \frac83\left(\sqrt3-1\right)$$

27) The equation of the inner circle is $r=2\cos\theta$

$$\cancel{\iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{2} r\,dr\,d\theta - 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \frac{16\pi}{3} - 2\int_0^{\pi/2}\frac83\cos^3\theta = \frac{16\pi}{3} - \frac{16}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/2}}$$

$$\iint_D \sqrt{x^2+y^2}\,dA = \int_0^2\int_0^{2\pi} r\,r\,d\theta\,dr - 2\int_0^{\pi/2}\int_0^{2\cos\theta} r\,r\,dr\,d\theta$$

$$= \frac{16\pi}{3} - 2\int_0^{\pi/2} \frac{1}{3}(2\cos\theta)^3\,d\theta$$

$$= \frac{16\pi}{3} - \frac{16}{3}\int_0^{\pi/2} \cos\theta\,(1-\sin^2\theta)\,d\theta$$

$$= \frac{16\pi}{3} - \frac{16}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_{\theta=0}^{\pi/2}$$

$$= \frac{16\pi}{3} - \frac{32}{9}$$

30) $\cancel{\int_0^{2\pi}\int_0^1 (4+r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\left[4r+\frac{1}{3}r^3\right]_{r=0}^{1} d\theta = \int_0^{2\pi}\frac{13}{3}\,d\theta = \frac{26\pi}{3}}$

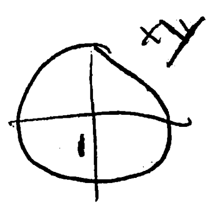
$$\int_0^{2\pi}\int_0^1\int_0^{4+r^2} z\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}\left[4+r^2\right]^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[16r + \frac{8r^4}{4} + \frac{r^6}{6}\right]_{r=0}^{1} d\theta$$

$$= \pi\left(16+4+\tfrac{1}{6}\right) = \frac{121}{6}\pi$$

34) $$\iiint_W z\sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} z r\; r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^2}{2}\left[(8-r^2)^2 - r^4\right] dr\,d\theta$$

38) 

$$\int_0^1\int_0^{2\pi}\int_0^4 f(r\cos\theta, r\sin\theta, z)\,r\,dz\,d\theta\,dr$$

Note that $0 \le z \le 3y$ only when $y \ge 0$.

44)

$$\int_0^2 \int_0^{\pi/2} \int_0^{3\sin\theta\, r} z^2\, r\, dz\, d\theta\, dr = \int_0^2 \int_0^{\pi/2} 9 \sin^3\theta\, r^4\, d\theta\, dr$$

$$= \int_0^2 9r^4 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\theta=0}^{\pi/2} dr = \int_0^2 6r^4\, dr = \frac{192}{5}$$

48) spherical

$$\cancel{\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} a}$$

$$\text{Volume} = \left|\int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta\right|$$

$$= \left|\int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^3}{3}\sin\phi\right]_{\rho=0}^{a} d\phi\, d\theta\right| = \left|\int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3}\sin\phi\, d\phi\, d\theta\right|$$

$$= \left|\int_0^{2\pi} \left[\frac{a^3\cos\phi}{3}\right]_{\phi=0}^{\pi} d\theta\right| = \left|\int_0^{2\pi} \frac{-2a^3}{3}\, d\theta\right| = \left|\frac{-4a^3}{3}\pi\right| = \frac{4a^3}{3}\pi$$

cylindrical

$$\text{Volume} = \int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{a} 2r\sqrt{a^2-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^{a} d\theta = \int_0^{2\pi} \frac{2}{3}a^3\, d\theta = \frac{4\pi}{3}a^3$$

50)

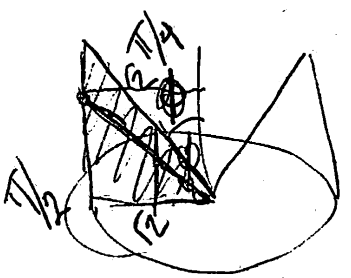
cylindrical

$$\int_0^{\sqrt{2}}\int_0^{2\pi}\int_0^{r} r^2\, r\,dz\,d\theta\,dr = \int_0^{\sqrt{2}}\int_0^{2\pi} r^4\,d\theta\,dr$$

$$= 2\pi\int_0^{\sqrt{2}} r^4\,dr = 2\pi\left[\frac{1}{5}r^5\right]_{r=0}^{\sqrt{2}}$$

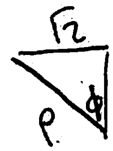
$$= \frac{8\sqrt{2}}{5}\pi$$

spherical



$$\int_{\pi/4}^{\pi/2}\int_0^{\frac{\sqrt{2}}{\sin\phi}}\int_0^{2\pi}\left[(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2\right]\rho^2\sin\phi\,d\theta\,d\rho\,d\phi$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{\frac{\sqrt{2}}{\sin\phi}}\int_0^{2\pi}\rho^4\sin^3\phi\,d\theta\,d\rho\,d\phi$$



$\sin\phi = \frac{\sqrt{2}}{\rho}$

$\rho = \frac{\sqrt{2}}{\sin\phi}$

$$= 2\pi\int_{\pi/4}^{\pi/2}\int_0^{\frac{\sqrt{2}}{\sin\phi}}\rho^4\sin^3\phi\,d\rho\,d\phi$$

$$= 2\pi\int_{\pi/4}^{\pi/2}\frac{1}{5}\,\frac{4\sqrt{2}}{\sin^5\phi}\sin^3\phi\,d\phi$$

$$= \frac{8\sqrt{2}}{5}\pi\left[-\cot(\phi)\right]_{\phi=\pi/4}^{\pi/2}$$

$$= \frac{8\sqrt{2}}{5}\pi$$

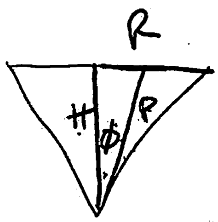
54)

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \left((\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2\right)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_{\phi=0}^{\pi} d\theta = \frac{4}{15}\int_0^{2\pi} d\theta = \frac{8\pi}{15}$$

60)



$$\cos\phi = \frac{H}{\rho} \qquad \rho = \frac{H}{\cos\phi}$$

$$\phi = \tan^{-1}\left(\frac{R}{H}\right)$$

$$\text{Volume} = \int_0^{\tan^{-1}(\frac{R}{H})}\int_0^{2\pi}\int_0^{\frac{H}{\cos\phi}} 1 \;\rho^2\sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\tan^{-1}(\frac{R}{H})}\int_0^{2\pi} \frac{H^3}{3\cos^3\phi}\sin\phi \, d\theta\, d\phi$$

$$= \int_0^{\tan^{-1}(\frac{R}{H})} 2\pi \frac{H^3}{3\cos^3\phi}\sin\phi \, d\phi$$

$$= \frac{2\pi H^3}{3}\left[\frac{1}{2}\tan^2\phi\right]_{\phi=0}^{\tan^{-1}(\frac{R}{H})} = \frac{2\pi H^3}{3}\cdot\frac{1}{2}\cdot\left(\frac{R}{H}\right)^2 = \frac{1}{3}\pi R^2 H$$