1. (5 points) Consider two vectors $\mathbf{v} = \langle 4, 2, -1\rangle$ and $\mathbf{w} = \langle 0, 1, 3\rangle$. If θ is the angle between $\mathbf{w}$ and $\mathbf{v}$, find $\sin(\theta)$ and $\cos(\theta)$.

Solution: First we note that $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{v}\|\cos(\theta)$ and that $\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\sin(\theta)$. Thus, to calculate $\cos(\theta)$ and $\sin(\theta)$, we calculate

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{4^2+2^2+(-1)^2} = \sqrt{21}\\
\|\mathbf{w}\| &= \sqrt{0^2+1^2+3^2} = \sqrt{10}\\
\mathbf{v}\cdot\mathbf{w} &= 0\cdot 4 + 2\cdot 1 + (-1)\cdot 3 = -1\\
\mathbf{v}\times\mathbf{w} &= \det\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 2 & -1\\ 0 & 1 & 3\end{vmatrix} = (2\cdot 3 - 1\cdot(-1))\,\mathbf{i} - (4\cdot 3 - 0\cdot(-1))\,\mathbf{j} + (4\cdot 1 - 0\cdot 2)\,\mathbf{k} = \langle 7, 12, 4\rangle\\
\|\mathbf{v}\times\mathbf{w}\| &= \sqrt{7^2+12^2+4^2} = 209.
\end{aligned}$$

Thus $\cos(\theta) = \frac{-1}{\sqrt{210}}$ and $\sin(\theta) = \frac{\sqrt{209}}{\sqrt{210}}$.

2. (5 points) Consider the vector valued function $\mathbf{r}(t) = e^{t^2-4}\mathbf{i} + t^3\mathbf{k}$. Does the tangent line at $t = 2$ pass through $(-3, 0, 2)$?

Solution: The tangent vector is $\mathbf{r}'(t) = \left\langle 2te^{t^2-4}, 0, 3t^2\right\rangle$ and so $\mathbf{r}'(2) = \langle 4, 0, 12\rangle$. Since $\mathbf{r}(2) = \langle 1, 0, 8\rangle$, the tangent line is $\ell(t) = \langle 1+4t, 0, 8+12t\rangle$. If the tangent line passed through $(3, 0, 2)$ then $1 + 4t = -3$ and $t = -1$. But $\ell(-1) = \langle -3, 0, -4\rangle$ and so the tangent line does not pass through $(-3, 0, 2)$.

3. (5 points) Consider the plane defined by the equation $x - 3y + 2z = 4$ and the plane defined by the normal vector $\langle 2, 0, 1\rangle$ through the point $(4, 0, 0)$. Find the intersection of the two planes. (Hint: the point $(4, 0, 0)$ is on both planes.)

Solution: The normal vector for the first plane is $\langle 1, -3, 2\rangle$, the two planes are note parallel, so their intersection is a line. Since

$$\begin{aligned}
\langle 1, -3, 2\rangle \times \langle 2, 0, 1\rangle &= \det\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & 2\\ 2 & 0 & 1\end{vmatrix}\\
&= (-3\cdot 1 - 0\cdot 2)\,\mathbf{i} - (1\cdot 1 - 2\cdot 2)\,\mathbf{j} + (1\cdot 0 - 2\cdot(-3))\,\mathbf{k}\\
&= \langle -3, 3, 6\rangle,
\end{aligned}$$

the vector $\langle -3, 3, 6\rangle$ lies in both planes. Then, since the point $(4, 0, 0)$ lies in both planes, the line of their intersection is defined by $\ell(t) = \langle 4-3t, 3t, 6t\rangle$.

4. (5 points) Show that the circumference of a circle of radius R is $2\pi R$. (Hint: Choose a "nice" parameterization of the circle and calculate the arclength)

Solution: We parameterize the circle via the vector valued function $\mathbf{r}(t) = \langle R\cos(t), R\sin(t)\rangle$. Then $\mathbf{r}'(t) = \langle -R\sin(t), R\cos(t)\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{R^2\sin(t) + R^2\cos(t)} = R$. The arclength is then $\int_0^{2\pi} \|\mathbf{r}'(t)\|\, dt = \int_0^{2\pi} R dt = 2\pi R$ and so the circle has circumference $2\pi R$.

5. (5 points) Let $\mathbf{r}(t) = \left\langle e^{-2t}, t^3 + 4t^2 - 2\right\rangle$. Decompose the acceleration vector into $a_T T(t) + a_N N(t)$ at $t = 0$.

Solution: We first calculate

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle -2e^{-2t}, 3t^2 + 8t\right\rangle \\
\mathbf{a}(t) &= \left\langle 4e^{-2t}, 6t + 8\right\rangle \\
\mathbf{v}(0) &= \langle -2, 0\rangle \\
\|\mathbf{r}(0)\| &= 2 \\
T(0) &= \frac{\mathbf{v}(0)}{\|\mathbf{v}(0)\|} = \langle -1, 0\rangle \\
\mathbf{a}(0) &= \langle 4, 8\rangle\,.
\end{aligned}$$

Then $a_T = a(0) \cdot T(0) = 4 \cdot (-1) + 0 \cdot 8 = -4$. Thus $a_N N(0) = a(0) - a_T T(0) = \langle 0, 8\rangle$. Since $N(0)$ is a unit vector, $N(0) = \langle 0, 1\rangle$ and $a_N = 8$.

6. (5 points) Suppose $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$. Explain why $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{w}) = 0$.

Solution: If $\mathbf{u}$ and $\mathbf{w}$ are parallel, then $\mathbf{u} \times \mathbf{w} = \mathbf{0}$ and so $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{w}) = \mathbf{w} \cdot \mathbf{0} = 0$. If $\mathbf{u}$ and $\mathbf{w}$ are not parallel, then the resulting vector is orthogonal to $\mathbf{u}$ and $\mathbf{w}$. But then since $\mathbf{w}$ and $\mathbf{u} \times \mathbf{w}$ are orthogonal, their dot product is 0.