1. (5 points) Evaluate the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy$ by a change of variables. Explain why your result implies that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Solution: We will integrate by changing to polar coordinates. Thus

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\ dx\,dy &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\ dr\,d\theta \\ &= \int_0^{2\pi} \lim_{a\longmapsto\infty}\int_0^a re^{-r^2} dr\,d\theta \\ &= \int_0^{2\pi} \lim_{a\longmapsto\infty}\left[-\frac{e^{-r^2}}{2}\right]_{r=0}^{a} d\theta \\ &= \int_0^{2\pi} \lim_{a\longmapsto\infty}\frac{1-e^{-a^2}}{2}\ d\theta \\ &= \int_0^{2\pi}\frac{1}{2}\ d\theta \\ &= \pi.\end{aligned}$$

We note that

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dydx = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2.$$

Thus $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$..

2. (5 points) Show that $(-5,0)$ is an extreme value of $f(x,y) = 4xe^{-y^2} + \sin(y)y^2$ along the ellipse given by $\left(\frac{x+2}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$.

Solution: Let $g(x,y) = \left(\frac{x+2}{3}\right)^2 + \left(\frac{y}{4}\right)^2 - 1$. By the method of Lagrange multipliers it suffices to show that there is some $\lambda \neq 0$ such that $\nabla f(-5,0) = \lambda\nabla g(-5,0)$ and that $g(-5,0) = 0$. First we observe that

$$g(-5,0) = \left(\frac{-5+2}{3}\right)^2 + \left(\frac{0}{4}\right)^2 - 1 = 0.$$

Further,

$$\begin{aligned}\nabla f(x,y) &= \left\langle 4e^{-y^2}, 8xye^{-y^2} + \cos(y)y^2 + 2\sin(y)y\right\rangle \\ \nabla g(x,y) &= \left\langle \frac{2(x+2)}{6}, \frac{2y}{16}\right\rangle.\end{aligned}$$

And so $\nabla f(-5,0) = \langle 4,0\rangle$ and $\nabla g(-5,0) = \langle -1,0\rangle$, and choosing $\lambda = -4$, yields $\nabla f(-5,0) = \lambda\nabla g(-5,0)$.

3. (5 points) Let $f(x,y) = (4y^2 - x^2)e^{-x^2-y^2}$. Using that

$$\nabla f = \left\langle e^{-x^2-y^2}(-2x)(1+4y^2-x^2), e^{-x^2-y^2}(-2y)(-4+4y^2-x^2)\right\rangle,$$

verify that $(\pm 1, 0)$, $(0,0)$, and $(0,\pm 1)$ are critical points of f. Use the fact that these five points are the only critical points of f to find the global extreme values of f on $x^2 + y^2 \leq 2$.

Solution: In order for (x,y) to be a critical point of f, either $\nabla f(x,y) = \langle 0,0\rangle$ or $\nabla f(x,y)$ is undefined. Now

$$\begin{aligned}\nabla f(0,0) &= \left\langle e^0(0)(1), e^0(0)(-4)\right\rangle = \langle 0,0\rangle \\ \nabla f(\pm 1,0) &= \left\langle e^{-1}(\mp 2)(0), e^{-1}(0)(-5)\right\rangle = \langle 0,0\rangle \\ \nabla f(0,\pm 1) &= \left\langle e^{-1}(0)(5), e^{-1}(\mp 2)(0)\right\rangle = \langle 0,0\rangle\end{aligned}$$

Thus the listed points are all critical points. Now the boundary is defined by $x^2 + y^2 = 2$, and so $x = \pm\sqrt{y^2-2}$. Thus along the boundary $f(x,y) = f(\pm\sqrt{2-y^2}, y) = (5y^2-2)e^{-2}$, which has derivative $10e^{-2}y$. This is zero when $y = 0$ and thus $x = \pm\sqrt{2}$. Now since the we have broken the boundary up into two regions, we also have to consider the end points of those regions, specifically $(0, \pm\sqrt{2})$. Considering the possible extreme points we have

$$\begin{aligned}f(0,0) &= 0 \\ f(\pm 1, 0) &= -e^{-1} \\ f(0,\pm 1) &= 4e^{-1} \quad \text{(maxima)} \\ f(\pm\sqrt{2}, 0) &= -2e^{-2} \quad \text{(minima)} \\ f(0, \pm\sqrt{2}) &= 8e^{-2}\end{aligned}$$

4. (5 points) The surface of the Martian terrain is given by the function $f(x,y) = e^{-y}x^2$. The Martian rover is facing "uphill" at the point $(-3,0)$ and can only handle an upward slope of at most $\frac{9\sqrt{2}}{2}$ and a downward slope of at most $9\sqrt{2}$. Describe the directions in which the rover can travel and describe the direction the rover can go without going up or down.

Solution: The slope in a unit direction u at a $(-3,0)$ is given by the directional derivative, $D_u(-3,0) = \nabla f(-3,0)\cdot u = \|\nabla f(-3,0)\|\,\|u\|\cos(\theta)$ where θ is the angle between the gradient and u. Now $\nabla f(x,y) = \left\langle 2xe^{-y}, -x^2e^{-y}\right\rangle$ and so $\nabla f(-3,0) = \langle -6,-9\rangle$ and $\|\nabla f(-3,0)\| = \sqrt{36+81} = 3\sqrt{4+9} = 3\sqrt{13}$. Since the rover is facing "uphill" (in the direction of the gradient), it can move at angle θ from its current heading such that $-\frac{3}{\sqrt{13}} \leq \cos(\theta) \leq \frac{3\sqrt{2}}{\sqrt{13}}$. The direction of the level curve is given by $\pm 1\frac{\langle 9,-6\rangle}{3\sqrt{13}} = \pm 1\left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right\rangle$.

5. (5 points) Find the average value of $f(x,y) = \sin(2x+3y) + x + y^2$ over the square $0 \le x \le \pi$ and $0 \le y \le \frac{2\pi}{3}$.

Solution: The value of $f(x,y)$ on the region is

$$\int_0^\pi \int_0^{\frac{2\pi}{3}} \sin(2x+3y) + x + y^2 dydx = \int_0^\pi \left[\frac{-1}{3}\cos(2x+3y) + xy + \frac{1}{3}y^3 \right]_{y=0}^{\frac{2\pi}{3}} dx$$
$$= \int_0^\pi \frac{-1}{3}\cos(2x+2\pi) + \frac{2\pi}{3}x + \frac{8\pi^3}{27} + \frac{1}{3}\cos(2x)dx$$
$$= \int_0^\pi \frac{2\pi}{3}x + \frac{8\pi^3}{27}dx$$
$$= \left[\frac{\pi}{3}x^2 + \frac{8\pi^3}{27}x \right]_{x=0}^{\pi}$$
$$= \frac{\pi^3}{3} + \frac{8\pi^4}{27}.$$

Since the area of the region is $\frac{2\pi}{3} \times \pi = \frac{2\pi^3}{3}$, this implies that the average value of f on the region is $\frac{\pi^2}{2} + \frac{4\pi^3}{9}$.

6. (5 points) An object lying in the positive octant is bounded above by $z = 3$ and below by $z = \sqrt{x^2+y^2}$ has density function $\rho(x,y,z) = xyz$. Set up the integral for the mass of the object in rectangular and cylindrical coordinates.

Solution: Since the object is in the positive octant, $x, y \ge 0$ and the projection of the object into the xy-plane is the quarter circle of radius 3 since this is when $3 = z = \sqrt{x^2+y^2}$. Thus in rectangular coordinates the integral is

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 xyz \ dz\, dy\, dz.$$

Converting this to cylindrical coordinates $r \le z \le 3$, $0 \le r \le 3$ and $0 \le \theta \le \frac{\pi}{2}$. Thus we have the integral

$$\int_0^3 \int_0^{\frac{\pi}{2}} \int_r^3 (r\cos(\theta))(r\sin(\theta))zr \ dz\, d\theta\, dr = \int_0^3 \int_0^{\frac{\pi}{2}} \int_r^3 r^3 z\cos(\theta)\sin(\theta) \ dz\, d\theta\, dr.$$