1. (10 points) Show that the matrix
$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$
has eigenvalues $3, 3, 0$ and find the eigenvectors associated with 3.

Solution:

$$\begin{aligned}
&\det\begin{pmatrix} 4-\lambda & -2 & 2 \\ 1 & 1-\lambda & 2 \\ -1 & 2 & 1-\lambda \end{pmatrix} \\
&= (4-\lambda)\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} - (-2)\det\begin{pmatrix} 1 & 2 \\ -1 & 1-\lambda \end{pmatrix} + 2\det\begin{pmatrix} 1 & 1-\lambda \\ -1 & 2 \end{pmatrix} \\
&= (4-\lambda)((1-\lambda)^2 - 4) + 2(1-\lambda+2) + 2(2+1-\lambda) \\
&= (4-\lambda)(-3-2\lambda+\lambda^2) + 12 - 4\lambda \\
&= -12 - 8\lambda + 4\lambda^2 + 3\lambda + 2\lambda^2 - \lambda^3 + 12 - 4\lambda \\
&= -\lambda^3 + 6\lambda^2 + 9\lambda \\
&= -\lambda(\lambda-3)^2.
\end{aligned}$$

Thus the eigenvalues are 3, 3, and 0. Perform Guassian elimination on $A - 3I$
$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the eigenvalues are
$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

2. Find the general form of the solution for each of the following differential equations.

(a) (5 points) $3y'' - y' - 2y = 0$

Solution: The characteristic polynomial is $3r^2 - r - 2 = 0$ which has roots $\frac{1\pm\sqrt{1+24}}{6} = \frac{1\pm 5}{6}$. Thus the general form of the solution is $c_1 e^t + c_2 e^{-\frac{2}{3}t}$.

(b) (5 points) $y'' + 4y' + 13y = 0$

Solution: The characteristic polynomial is $r^2+4r+13 = 0$ which has roots $\frac{-4\pm\sqrt{16-52}}{2} = \frac{-4\pm 6i}{2} = -2 \pm 3i$. Thus the general form of the solution is $e^{3t}(c_1\cos(2t) + c_2\sin(2t))$.

(c) (5 points) $4y'' + 4y' + y = 0$.

Solution: The characteristic polynomial is $4r^2+4r+1=0$ which has roots $\frac{-4\pm\sqrt{16-16}}{8} = -\frac{1}{2}$. Thus the general form of the solution is $c_1e^{-\frac{t}{2}} + c_2te^{-\frac{t}{2}}$.

3. Find the general form of the solution for each of the following systems of differential equations of the form $x' = Ax$.

(a) (15 points) $A = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}$

Solution:

$$\det\begin{pmatrix} -1-\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = (1+\lambda)(3+\lambda)+2 = \lambda^2+4\lambda+5.$$

Thus the eigenvalues are $-2 \pm i$ and the eigenvectors are $\begin{bmatrix} 1 \\ 2 \mp i \end{bmatrix}$. Thus the general form of the solution is

$$c_1e^{-2t}\begin{bmatrix} \cos(t) \\ 2\cos(t)-\sin(t) \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} \sin(t) \\ 2\sin(t)+\cos(t) \end{bmatrix}$$

.

(b) (15 points) $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

Solution:

$$\det\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)-12 = \lambda^2-3\lambda-10 = (\lambda-5)(\lambda+2).$$

Thus the eigenvalues are 5 and -2 and the eigenvectors are $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Thus the general form of the solution is

$$c_1e^{5t}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(c) (20 points) $A = \begin{bmatrix} 0 & -3 \\ 3 & -6 \end{bmatrix}$

Solution:

$$\det\begin{pmatrix} -\lambda & -3 \\ 3 & -6-\lambda \end{pmatrix} = \lambda(6+\lambda)+9 = (\lambda+3)^2.$$

Thus the eigenvalue is -3 with multiplicity two and the associated eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The generalized eigenvector is a solution to

$$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and so $\xi = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$. The general solution is then

$$(c_2 t + c_1)e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}.$$

4. (15 points) Find the Laplace transform of y satisfying $y'' - 5y' + 3y = \frac{t-2\pi}{2\pi} u_\pi(t) + \sin(t)u_{2\pi} + e^{4t}\cos(3t)$ and $y(0) = y'(0) = 0$.

Solution: The Laplace transform of the left hand side of the equation is $(s^2 - 5s + 3)\mathcal{L}(y)$ since $y(0) = y'(0) = 0$. Noting that $t - 2\pi = t - \pi - \pi$ and that $\sin(t) = \sin(t - 2\pi)$, the Laplace transform of the right hand side is $\frac{e^{-\pi s}}{2\pi s^2} - \frac{e^{-\pi s}}{2s} + \frac{e^{-2\pi s}}{s^2+1} + \frac{s-4}{(s-4)^2+3^2}$. Combining these results we get that

$$\mathcal{L}(y) = \frac{e^{-\pi s} - \pi s e^{-\pi s}}{2\pi s^2(s^2 - 5s + 3)} + \frac{e^{-2\pi s}}{(s^2+1)(s^2 - 5s + 3)} + \frac{s-4}{(s - 8s + 25)(s^2 - 5s + 3)}$$

5. (15 points) Find y if $\mathcal{L}(y) = e^{-3s}\frac{2s^2+9s-9}{s^2(s^2+9)} + \frac{4}{s-2}$.

Solution: First consider the partial fraction decomposition of $\frac{2s^2+9s-9}{s^2(s^2+9)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9}$. Simplifying and equating like terms, we have that

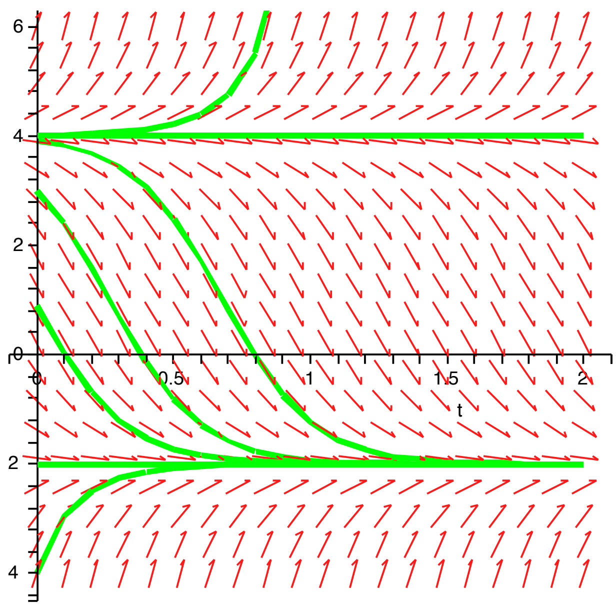
$$\begin{aligned} A + C &= 0 \\ B + D &= 2 \\ 9A &= 9 \\ 9B &= -9. \end{aligned}$$

Thus $A = 1$, $B = -1$, $C = -1$, and $D = 3$ and $\mathcal{L}^{-1}(\frac{2s^2+9s-9}{s^2(s^2+9)}) = 1 - t - \cos(3t) + \sin(2t)$. Noting that $\mathcal{L}^{-1}(\frac{4}{s-2}) = 4e^{2t}$ we have that

$$y = (1 - (t-3) - \cos(3t - 9) + \sin(3t - 9))\, u_3(t) + 4e^{2t}.$$

6. (10 points) Consider the differential equation $y' = (y + 2)(y - 4)$, plot a set of representative solution curves. Note that $(y + 2)(y - 4)$ achieves its minimum at $y = 1$.

Solution: Note first that $y' > 0$ for $y > 4$ and $y < -2$, and $y < 0$ for $-2 < y < 4$. Further y is concave up when $-2 < y < 1$ and $y > 4$ and concave down when $y < -2$ and $1 < y < 4$. Thus we have the following representative solutions with $y = -2$ being a stable solution and $y = 4$ being unstable.



7. (10 points) Write the following differential equation as a system of first order differential equations,

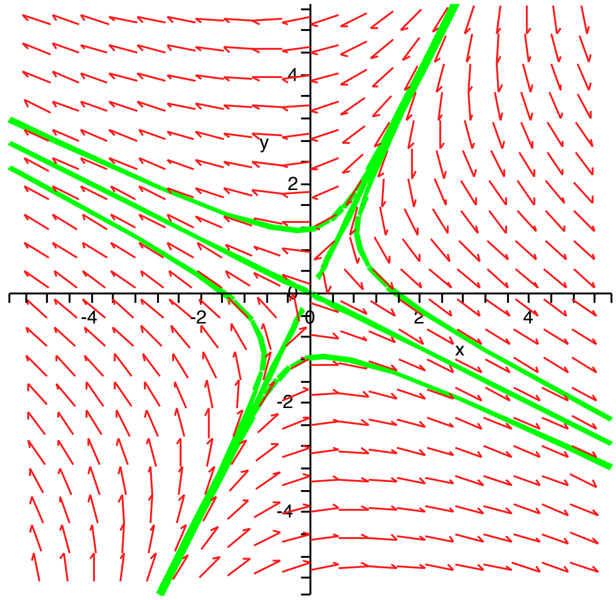
$$y''' + 3y' + y = \cos(t)$$

Solution: Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then the differential equation can be defined by

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -3x_2 - x_1 + \cos(t). \end{aligned}$$

8. (15 points) Plot a set of representative solutions curves for the system of differential equations $x' = Ax$, where A has eigenvalues -2 and 5 with associated eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, respectively.

Solution:



9. (20 points) Consider the following situation: There are two tanks of salt water. The first tank starts with 300 gallons and 50 pounds of salt and the second tank starts with 200 gallons of water and no salt. The first tank has a 1 pound per gallon solution flowing into it at a rate of 6 gallons a minute. Two gallons per minute from tank one leave the system while 4 gallons per minute flow from the first tank to the second tank. The second tank has a 3 gallons a minute of fresh water flowing into it and 7 gallons per minute flowing out of the system from tank two. Write a system of differential equations that governs the pounds of salt in each of the tanks. (Note: The volume of liquid in each tank stays constant as a function of time)

Solution: Let p_1 be the pounds of salt in the first tank and let p_2 be the pounds of salt in the second tank. The rate salt flows into the first tank is $6\frac{\text{lbs}}{\text{gal}} \times 1\frac{\text{gal}}{\text{min}}$ the rate salt flows out of the first tank is $6\frac{\text{gal}}{\text{min}} \times \frac{p_1\text{lbs}}{300\text{gal}}$. The rate salt flows into the second tank is $3\frac{\text{gal}}{\text{min}} \times 0\frac{\text{lbs}}{\text{gal}} + 4\frac{\text{gal}}{\text{min}} \times \frac{p_1\text{lbs}}{300\text{gal}}$. Salt flows out of the second tank at a rate of $7\frac{\text{gal}}{\text{min}} \times \frac{p_2\text{lbs}}{200\text{gal}}$. Thus the system is governed by the differential equation

$$p_1' = 6 - \frac{p_1}{50}$$
$$p_2' = \frac{p_1}{75} - \frac{7p_2}{200}$$

with the initial conditions $p_1(0) = 50$ and $p_2(0) = 0$.

10. (15 points) Give two single variable differential equations whose solution can be used to solve for x and explain. $x' = Ax + \begin{bmatrix} e^{4t} \\ 3e^{-2t} \end{bmatrix}$ where $A = TDT^{-1}$ with

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad T = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

Solution: Let $x = Ty$ and then $x' = Ty'$. Substituting into the differential equation we get $Ty' = TDT^{-1}Ty + \begin{bmatrix} e^{4t} \\ 3e^{-2t} \end{bmatrix}$ Then simplifying and multiplying on the left by T^{-1} we have $y' = Dy + T^{-1}\begin{bmatrix} e^{4t} \\ 3e^{-2t} \end{bmatrix}$. In otherwords, $y_1' = 3y_1 + 2e^{4t} - 15e^{-2t}$ and $y_2' = -2y_2 - e^{4t} + 9e^{-2t}$. By solving these two differential equations, we get the vector solution y, but then $x = Ty$ and so we have a solution to the original system of differential equations.

11. Consider the differential equation $x' = Ax + \begin{bmatrix} e^{4t} \\ 5e^{-2t} \end{bmatrix}$ where the solutions to the homogeneous is $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t}$.

   (a) (15 points) Use variation of parameters to find a particular solution to the above differential equation.

Solution: The fundamental matrix is $\Psi = \begin{bmatrix} e^{3t} & -2e^{-2t} \\ 2e^{3t} & e^{2t} \end{bmatrix}$. By the method of variation of parameters we want to find a solution to $\Psi \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} e^{4t} \\ 5e^{-2t} \end{bmatrix}$. Performing Gaussian elimination to find u_1' and u_2', we get

$$\begin{bmatrix} e^{3t} & -2e^{-2t} & e^{4t} \\ 2e^{3t} & e^{-2t} & 5e^{-2t} \end{bmatrix} \to \begin{bmatrix} e^{3t} & -2e^{-2t} & e^{4t} \\ 0 & 5e^{-2t} & 5e^{-2t} - 2e^{4t} \end{bmatrix} \to \begin{bmatrix} e^{3t} & -2e^{-2t} & e^{4t} \\ 0 & 1 & 1 - \frac{2}{5}e^{6t} \end{bmatrix}.$$

Thus $u_2' = 1 - \frac{2}{5}e^{6t}$ and $u_1' = 2e^{-5t} + \frac{1}{5}e^{t}$. And hence $u_2 = t - \frac{2}{30}e^{6t} + c_2$ and $u_1 = \frac{-2}{5}e^{-5t} + \frac{1}{5}e^{t} + c_1$ and

$$x = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \frac{e^{-2t}}{5} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{e^{4t}}{5} + c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} -2t \\ t \end{bmatrix} e^{-2t} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{e^{4t}}{15} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t}$$

   (b) (10 points) Use the method of undetermined coefficients to set up a system of equations that will yield a particular solution to the above differential equation. (Note: Express the system in terms of the matrix A.)

Solution: Based on the nonhomogenous portion of the differential equation, our particular guess is $\vec{a}e^{4t} + (\vec{b}t + \vec{c})e^{-2t}$. Substituting we have that

$$4\vec{a}e^{4t} + (\vec{b} - 2\vec{c} - 2\vec{b}t)e^{-2t} = A\vec{a}e^{4t} + A(\vec{b}t + \vec{c})e^{-2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} e^{-2t}.$$

This yields the following systems of equations

$$\begin{aligned} 4\vec{a} &= A\vec{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vec{b} - 2\vec{c} &= A\vec{c} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} \\ -2\vec{b} &= A\vec{b}. \end{aligned}$$