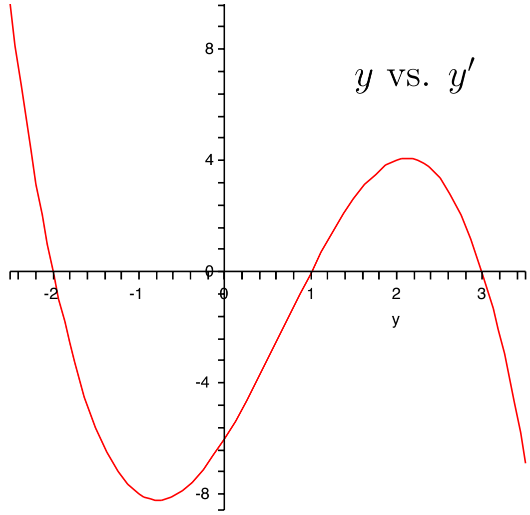
1. (20 points) Consider the differential equation

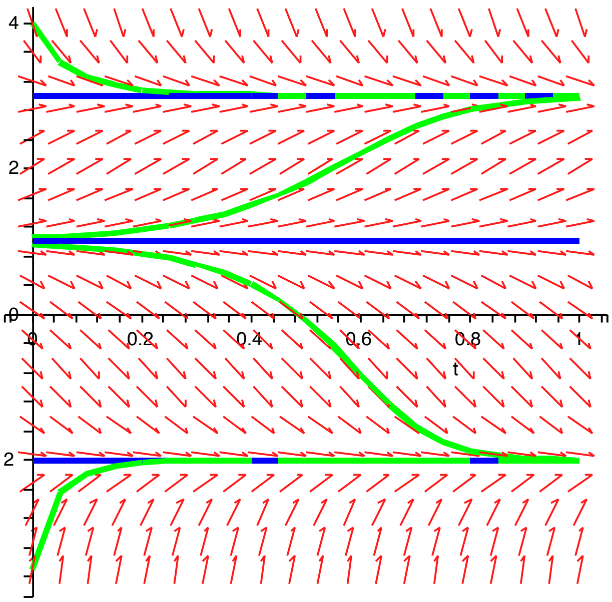
$$y' = (1-y)(y+2)(y-3).$$

Plot a representative set of solutions curves. Be sure to draw any equilibrium solutions and label their stability.



Solution: Since $y' = 0$ when $y = -2, 1, 3$ these are the equilibrium solutions. When $y < -2$, $y' > 0$, when $-2 < y < 1$, $y' < 0$, when $1 < y < 3$, $y' > 0$ and when $y > 3$ $y' < 0$. Thus $y = -2$ is stable, $y = 1$ is unstable, and $y = 3$ is stable.

Let r_1 be the local minimum between -2 and 1 and let r_2 be the local minimum between 1 and 3. The solution is concave up on $(-2, r_1) \cup (1, r_2) \cup (3, \infty)$ and concave down on $(-\infty, -2) \cup (r_1, 1) \cup (r_2, 3)$.



2. (20 points) A tank starts off with 200 gallons of water and 25 pounds of salt in it. A fresh water solutions is pumped into the tank at a rate of $\sin(t) + 5$ gallons per minute, the tank is being emptied at a rate of $5 - \sin(t)$ gallons per minute. Set up, **BUT DO NOT SOLVE**, the differential equation for the pounds of salt in the tank at time t.

Solution: Let $P(t)$ be the pounds of salt in the tank at time t and let $G(t)$ be the gallons of water in the tank at time t. Note $P(0) = 25$ and $G(0) = 200$. Since there is $\sin(t) + 5$ gallons being pumped into the tank and $5 - \sin(t)$ being pumped out,

$$G(t) - G(0) = \int_0^t \sin(s) + 5 - (5 - \sin(s))ds = \int_0^t 2\sin(s)ds = -2\cos(s)|_{s=0}^t = 2 - 2\cos(t).$$

Thus $G(t) = 202 - 2\cos(t)$. Now the salt is coming in at a rate of 0 since it is a freshwater solution and it is leaving at $\frac{P(t)}{G(t)}(5 - \sin(t))$ thus $P(t)$ satisfies

$$\frac{dP}{dt} = \frac{\sin(t) - 5}{202 - 2\cos(t)} P(t),$$

with $P(0) = 25$.

3. Find the general **REAL** solutions for the following differential equations.
   (a) (5 points) $y'' - 6y' + 13y = 0$
   (b) (5 points) $y'' - 5y' + 4y = 0$
   (c) (5 points) $y'' + 2y' + y = 0$

Solution:

(a) The characteristic equation for $y'' - 6y' + 13y = 0$ is $r^2 - 6r + 13 = 0$ which has roots $\frac{6\pm\sqrt{36-4*13}}{2} = 3\pm 2i$. Thus the general form of the solution is $e^{3t}(c_1\cos(2t) + c_2\sin(2t))$.

(b) The characteristic equation for $y'' - 5y' + 4y = 0$ is $r^2 - 5r + 4 = 0$ which has roots 4 and 1. Thus the general form of the solution is $c_1e^{4t} + c_2e^t$.

(c) The characteristic equation for $y'' + 2y' + y = 0$ is $r^2 + 2r + 1 = 0$ which has roots $-1, -1$. Thus the general form of the solution is $c_1e^{-t} + c_2te^{-t}$.

4. (10 points) Find an appropriate integrating factor and solve $y' - 2ty = e^{t^2+t}$ with the initial condition that $y(0) = 1$.

Solution: Since y is multiplied by $-2t$ the integrating factor should be $e^{\int -2tdt} = e^{-t^2}$. Multiplying through we have that $e^{-t^2}y' - 2te^{-t^2} = e^t$. Thus, integrating we have $e^{-t^2}y = e^t + C$. Since $y(0) = 1$, we have $C = 0$ and $y = e^{t+t^2}$.

5. (10 points) Find a general solution to the differential equation $y' = \frac{e^t}{\cos(y)}$.

Solution: We note that is a separable differential equation, corresponding to $\cos(y)dy = e^t dt$. Integrating, we get $\sin(y) = e^t + C$ which implicitly defines the solution as $\sin(y) - e^t = C$.

6. (10 points) Suppose that $y_1 = \cos(2t)$ and $y_2 = \sin(5t) - 4t$ satisfy the differential equation $y'' + p(t)y' + q(t)y = 0$, find a solution $y(t)$ with $y(0) = 4$ and $y'(0) = -2$. (Hint: $y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1$.)

Solution: Since $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, and $y_2'(0) = 1$, y_1 and y_2 form a fundamental set of solutions and thus the solution through the initial conditions is $4\cos(2t) - 2(\sin(5t) - 4t)$.

7. (15 points) Use the integrating factor $x\cos(x)$ to solve the following differential equation,

$$\frac{2(y+1)}{\cos(x)} + \frac{e^y + 1}{x} + \left(\frac{x}{\cos(x)} + \frac{e^y \tan(x)}{x}\right) y' = 0.$$

Solution: Multiplying through by the integrating factor we have that

$$2x(y+1) + (e^y + 1)\cos(x) + \left(x^2 + e^y \sin(x)\right) y' = 0.$$

Let $M(x,y) = 2x(y+1) + (e^y+1)\cos(x)$ and $N(x,y) = x^2 + e^y \sin(x)$. Note that $M_y(x,y) = 2x + \cos(x)e^y$ and $N_x(x,y) = 2x + \cos(x)e^y$ and thus $M_y(x,y) = N_x(x,y)$ and so the resulting equation is exact and so the solution is implicitly defined by $\Psi(x,y) = C$, where $\Psi_x = M(x,y)$ and $\Psi_y = N(x,y)$. Thus

$$\Psi(x,y) = \int 2x(y+1) + (e^y+1)\cos(x)dx = x^2(y+1) + (e^y+1)\sin(x) + h(y).$$

But $N(x,y) = \Psi_y(x,y) = x^2 + e^y \sin(x) + h'(y)$, implying that $h'(y) = 0$ and so $\Psi(x,y) = x^2(y+1) + (e^y+1)\sin(x)$ and the differential equation is solved implicitly by $x^2(y+1) + (e^y+1)\sin(x) = C$.