1. (15 points) Find all the eigenvalues (with multiplicity) and the eigenvector(s) associated with the largest eigenvalue for the following matrix.

$$\begin{bmatrix} -6 & 3 & -6 \\ 2 & -1 & 2 \\ 4 & -2 & 4 \end{bmatrix}$$

Solution:

$$\begin{aligned}
&\det \begin{pmatrix} -6-\lambda & 3 & -6 \\ 2 & -1-\lambda & 2 \\ 4 & -2 & 4-\lambda \end{pmatrix} \\
&= (-6-\lambda)\det\begin{pmatrix} -1-\lambda & 2 \\ -2 & 4-\lambda \end{pmatrix} - 3\det\begin{pmatrix} 2 & 2 \\ 4 & 4-\lambda \end{pmatrix} + (-6)\det\begin{pmatrix} 2 & -1-\lambda \\ 4 & -2 \end{pmatrix} \\
&= (-6-\lambda)(-(1+\lambda)(4-\lambda)+4) - 3(8-2\lambda-8) - 6(-4+4+4\lambda) \\
&= (-6-\lambda)(-4-3\lambda+\lambda^2+4) + 6\lambda - 24\lambda \\
&= \lambda\left((-6-\lambda)(\lambda-3) - 18\right) \\
&= \lambda\left(18 - 3\lambda - \lambda^2 - 18\right) \\
&= -\lambda^2(3+\lambda).
\end{aligned}$$

Thus the eigenvalues are -3, 0, and 0, with 0 being the largest. Now reducing the matrix associated with the eigenvalue 0, we get

$$\begin{bmatrix} -6 & 3 & -6 \\ 2 & -1 & 2 \\ 4 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, two linearly independent eigenvectors associated with 0 are

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

2. (20 points) The matrix A has the eigenvalues $3 \pm 2i$, -1, and 2. The eigenvector are

$$\begin{bmatrix} -1 \\ -4 \\ 2 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -2 \\ 1 \\ 4 \\ -2 \end{bmatrix},$$

respectively. Write down the **real** solution to the differential equation $\mathbf{x}' = A\mathbf{x}$ with

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 3 \\ 14 \\ -6 \end{bmatrix}.$$

Solution: The general form of the solution is

$$c_1 e^{3t}\begin{bmatrix}-\cos(2t)\\-4\cos(2t)+\sin(2t)\\2\cos(2t)\\-2\sin(2t)\end{bmatrix}+c_2e^{3t}\begin{bmatrix}-\sin(2t)\\-4\sin(2t)-\cos(2t)\\2\sin(2t)\\2\cos(2t)\end{bmatrix}+c_3\begin{bmatrix}2\\0\\3\\0\end{bmatrix}e^{-t}+c_4\begin{bmatrix}-2\\1\\4\\-2\end{bmatrix}e^{2t}.$$

Thus we are looking for a solution to

$$\begin{bmatrix}-1&0&2&-2\\-4&-1&0&1\\2&0&3&4\\0&2&0&-2\end{bmatrix}\vec{\mathbf{c}}=\begin{bmatrix}0\\3\\14\\-6\end{bmatrix}.$$

Applying Gaussian elimination

$$\begin{bmatrix}-1&0&2&-2&0\\-4&-1&0&1&3\\2&0&3&4&14\\0&2&0&-2&-6\end{bmatrix}\to\begin{bmatrix}-1&0&2&-2&0\\0&-1&-8&9&3\\0&0&7&0&14\\0&2&0&-2&-6\end{bmatrix}\to\begin{bmatrix}-1&0&2&-2&0\\0&-1&-8&9&3\\0&0&7&0&14\\0&0&0&16&0\end{bmatrix},$$

and thus $c_4=0, c_3=2, c_2=-19, c_1=0$. The solution is then

$$-19e^{3t}\begin{bmatrix}-\sin(2t)\\-4\sin(2t)-\cos(2t)\\2\sin(2t)\\2\cos(2t)\end{bmatrix}+2\begin{bmatrix}2\\0\\3\\0\end{bmatrix}e^{-t}$$

3. (20 points) Find the Laplace transform of y for the following differential equation

$$3y''+y'-y=g(t),\quad y(0)=1,\quad y'(0)=-2$$

where

$$g(t)=\begin{cases}0 & t\le 3\\(t-3)/6 & 3<t\le 6\\e^{4(t-6)}\sin(2(t-6)) & 6<t\end{cases}.$$

Solution: First we note that $g(t)=\frac{t-3}{6}(u_3(t)-u_6(t))+e^{4(t-6)}\sin(2(t-6))u_6(t)=\frac{t-3}{6}u_3(t)+\left(e^{4(t-6)}\sin(2(t-6))-\frac{t-6+3}{6}\right)u_6(t)$ and thus $\mathcal{L}(g(t))=\frac{e^{-3s}}{6s^2}+\frac{2e^{-6s}}{(s-4)^2+4}-\frac{e^{-6s}}{6s^2}+\frac{e^{-6s}}{2s}$. We also have $\mathcal{L}(3y''+y'-y)=3s^2\mathcal{L}(y)-3sy(0)-3y'(0)+s\mathcal{L}(y)-y(0)-\mathcal{L}(y)$. Combining these we have that

$$\mathcal{L}(y)=\frac{e^{-3s}-e^{-6s}}{6s^2(3s^2+s-1)}+\frac{2e^{-6s}}{((s-4)^2+4)(3s^2+s-1)}+\frac{e^{-6s}}{2s(3s^2+s-1)}-\frac{3s+5}{3s^2+s-1}.$$

4. (15 points) Given that $\mathcal{L}\{y\} = \frac{4s^2+9}{s^2(s^2+9)}\left(e^{4s} - e^{8s}\right)$ find y.

Solution: Let $\frac{4s^2+9}{s^2(s^2+9)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9}$, then $(As+B)(s^2+9) + (Cs+D)s^2 = 4s^2+9$. Thus

$$\begin{aligned} 0 &= A + C \\ 4 &= B + D \\ 0 &= 9A \\ 9 &= 9B. \end{aligned}$$

Thus $B = 1$, $A = 0$, $D = 3$ and $C = 0$. Thus $\mathcal{L}(y) = \frac{3}{s^2+9} + \frac{1}{s^2}$ and $y = \sin(3t) + t$.

5. (15 points) Consider the differential equation $y'' + 4y' + 4y = e^{3t} + \sin 2t$ which has a homogeneous solution of $c_1e^{-2t} + c_2te^{-2t}$, set up, **but do not solve**, a system of linear equations to find the particular solution. (Hint: Use the method of undetermined coefficients.)

Solution: Based on the nonhomogeneous portion, we guess that the particular solution has the form $Ae^{3t}+B\cos(2t)+C\sin(2t)$, which has first derivative $3Ae^{3t}-2B\sin(2t)+2C\cos(2t)$ and second derivative $9Ae^{3t}-4B\sin(2t)-4C\cos(2t)$. Plugging into the differential equation we have

$$25Ae^{3t} - 8C\cos(2t) - 8B\sin(2t) = e^{3t} + \sin(2t).$$

Thus the desired system of equations is

$$\begin{aligned} 25A &= 1 \\ -8C &= 0 \\ -8B &= 1. \end{aligned}$$

6. (15 points) Given that $y(0) = 3$ and $y'(0) = -2$, find $y'''(0)$ for the solution to the differential equation

$$4y'' + 4t^2y = \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k.$$

(Hint: Find the series solution for the differntial equation about $t = 0$)

Solution: Let $y(t) = \sum_{n=0}^{\infty} a_nt^n$ and then $y'(t) = \sum_{n=1}^{\infty} na_nt^{n-1}$, $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_nt^{n-2}$, and $y'''(t) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_nt^{n-3}$. Thus $y'''(0) = 6a_3$. Now plug y into the differential equation

$$4\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 4t^2 \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} t^n$$

$$\sum_{j=0}^{\infty} 4(j+2)(j+1)a_{j+2}t^j + \sum_{j=2}^{\infty} 4a_{j-2}t^j = \sum_{j=0}^{\infty} t^j$$

$$8a_2 + 24a_3 t + \sum_{j=2}^{\infty} \left(4(j+2)(j+1)a_{j+2} + 4a_{j-2}\right) t^j = \sum_{j=0}^{\infty} t^j.$$

Thus $24a_3 = 1$ and so $y'''(0) = \frac{1}{4}$.