1. (5 points) Find the general (implicit) form for the solution to $y' = \sec(y)t^2$.

$$\frac{dy}{dt} = \sec(y)\, t^2$$

seperating variables we have

$$\cos(y)\,dy = \frac{dy}{\sec(y)} = t^2\,dt$$

integrating

$$-\sin(y) = \frac{1}{3}t^3 + C$$

which is an implicit form for the solution.

2. (5 points) A model rocket has two engines that produce 200 $\mathrm{kg\cdot m/s^2}$ of thrust each for 20 seconds. Unfortunately one of the engines fails after 10 seconds. Find the speed of the rocket after 20 seconds given that the constant for force exerted by air resistance is 10 $\mathrm{kg/s}$ and the rocket weighs 2 kg. For ease of calculation, treat the acceleration due to gravity as a constant 10 m/s^2.

By balance of forces

$$\underbrace{m}_{\text{mass of rocket}}\overbrace{a}^{\text{accelerating}} = F = -m\cdot \underbrace{g}_{\text{gravity}} + \overbrace{T}^{\text{rocket thrust}} - \underbrace{k}_{\text{air resistance constant}}\underbrace{v}_{\text{velocity of rocket}}$$

Thus $\dfrac{dv}{dt} = -10 + \dfrac{T}{2} - 5v = -5\left(v + 2 - \dfrac{T}{10}\right)$

Separating variables and solving we have

$$\left\{v + 2 - \frac{T}{10}\right\} = C_0 e^{-5t}$$

Now for the first 10s, $T = 2\cdot 200$ and $v(0) = 0$

so $v - 38 = C_0 e^{-5\cdot 0}$, $C_0 = -38$

After 10s, the speed is then $38(1-e^{-50})$. This forms the initial velocity for the second 10s when $T = 200$.

Thus $v(0) = 38(1-e^{-50})$ and

$$v - 18 = C_0' e^{-5t} \quad \text{so} \quad C_0' = 20 - 38e^{-50}$$

and the final rocket speed is $v(t) = 18 + (20 - 38e^{-50})e^{-50}$

$= 18 + 20e^{-50} - 38e^{-100}$.

3. (5 points) Find the solution for the differential equation $y' + \frac{3y}{t} = t^{-2}$ where $y(1) = 3$.

We will proceed by finding an integrating factor.

We want $\mu(t)$ so that

$$\mu(t)y' + \frac{3}{t}\mu(t)y = \frac{d}{dt}[\mu(t)y]$$

so $$\frac{3}{t}\mu = \frac{d}{dt}\mu$$

Separating variables we get

$$\frac{d\mu}{\mu} = \frac{3}{t}dt$$

Integrating

$$\ln|\mu| = \ln(t^3) + C$$

so $\mu = t^3$ works as an integrating factor.

Rewriting we have $\frac{d}{dt}[t^3 y] = t$

Integrating we get $t^3 y = \frac{1}{2}t^2 + C$, so

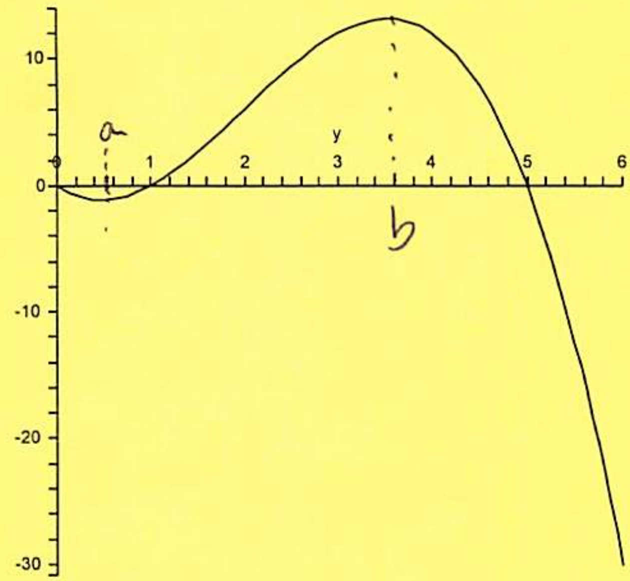
$$y(t) = \frac{1}{2t} + \frac{C}{t^3}$$

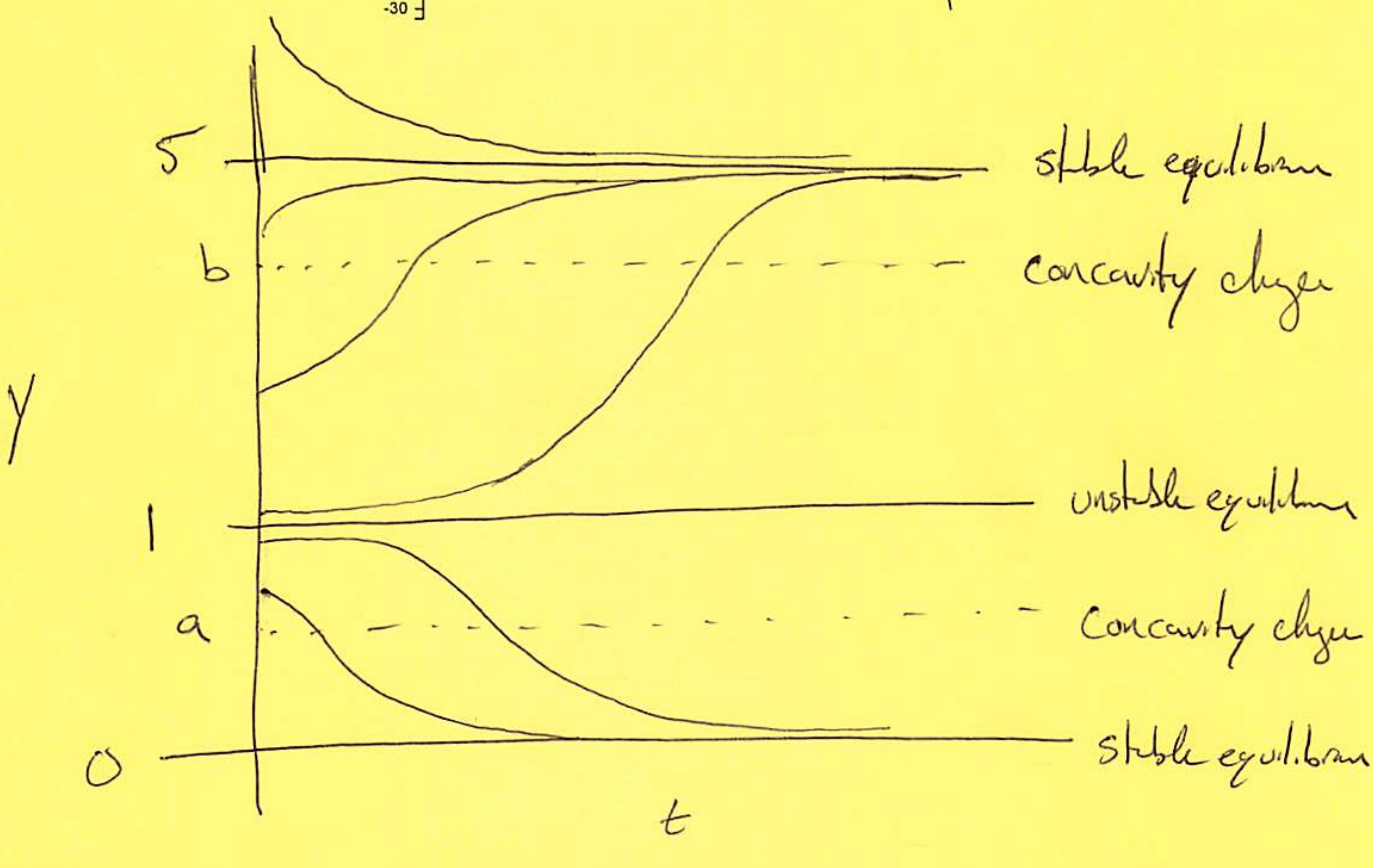
Since $y(1) = 3$, $3 = \frac{1}{2} + C$ and $C = 5/2$.

The final solution is $y(t) = \frac{1}{2t} + \frac{5}{2t^3}$.

 Name:________ Points earned:______

4. (5 points) Draw a representative set of solution curves for the differential equation $y' = f(y)$, where the plot of $f(y)$ is given below. Be sure to label all important points.





 Name:________ Points earned:______

5. (5 points) Find the general solution to

$$\underbrace{2x\sin(xy) + x^2y\cos(xy) + e^x\cos(y)}_{M} + \underbrace{\left(x^3\cos(xy) - e^x\cos(y)\right)}_{N} y' = 0$$

$$M_y = 2x^2\cos(xy) + x^2\cos(xy) - x^3y\sin(xy) - e^x\sin(y)$$

$$N_x = 3x^2\cos(xy) - x^3y\sin(xy) - e^x\cos(y)$$

~~So the two~~ So the differential equation is not exact.

6. (5 points) The half-life of titanium-44 is 63 years. Set up and solve a differential equation for amount of a 100 g sample present as a function of time. Use this to determine how long until there is 1 g of the sample left.

Let $T \equiv$ grams of titanium at time t.

Now $\frac{\partial T}{\partial t} = rT$ which has solution

$$T = C_0 e^{rt}.$$

Since the half life is 63 years we have

$$\frac{1}{2} \cancel{C_0} \cancel{e^{r\cdot 0}}^{\,1} = \cancel{C_0} e^{r\cdot 63}$$

$$\ln\left(\tfrac{1}{2}\right) = 63r \qquad \text{so } r = \frac{\ln\left(\frac{1}{2}\right)}{63}$$

Since $T(0) = 100 = C_0 e^{\frac{\ln(\frac{1}{2})}{63}\cdot 0} = C_0$, $C_0 = 100$

and $T(t) = 100 e^{\frac{\ln(\frac{1}{2})}{63} t}$

Thus there is 1 g left when

$$1 = 100 e^{\frac{\ln(\frac{1}{2})}{63} t}$$

$$\frac{1}{100} = e^{\frac{\ln(\frac{1}{2})}{63} t}$$

$$\ln\left(\frac{1}{100}\right) = \frac{\ln(\frac{1}{2})}{63} t$$

$$t = \frac{63 \ln\left(\frac{1}{100}\right)}{\ln\left(\frac{1}{2}\right)} = \frac{63 \ln(100)}{\ln(2)}.$$