1. (5 points) t and e^{t^2} are solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$, show that they are fundamental solutions and use this work to determine $p(t)$.

Solution: The Wronskian of t and e^{t^2} is $t \cdot 2te^{t^2} - 1 \cdot e^{t^2} = (2t^2 - 1)e^{t^2}$ which is non-zero for $t \neq \pm\frac{1}{\sqrt{2}}$, thus these form a fundamental solution on all intervals not including these points. To determine $p(t)$, we note that

$$\begin{aligned} W &= ce^{-\int p(t)} \\ (2t^2 - 1)e^{t^2} &= ce^{-\int p(t)} \\ \ln((2t^2 - 1)e^{t^2}) &= \ln(ce^{-\int p(t)}) \\ \ln(2t^2 - 1) + t^2 &= \ln(c) - \int p(t). \end{aligned}$$

Taking the derivative of both sides with respect to t we get,

$$\frac{4t}{2t^2 - 1} + 2t = -p(t)$$

so $p(t) = \frac{4t}{1-2t^2} - 2t$.

2. (5 points) Find all eigenvalues and the eigenvector corresponding to the smallest eigenvalue for

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: We first consider

$$\det \begin{bmatrix} -\lambda & 2 & 1 \\ 2 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

This is

$$\begin{aligned} &(-\lambda)(-\lambda)(1-\lambda) + 2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot 1 - (1 \cdot (-\lambda)1) - (1 \cdot 1 \cdot (-\lambda)) - ((1-\lambda) \cdot 2 \cdot 2) \\ &= -\lambda^3 + \lambda^2 + 4 + \lambda + \lambda - 4 + 4\lambda \\ &= -\lambda^3 + \lambda^2 + 6\lambda. \end{aligned}$$

Thus the eigenvalues are -2, 0, and 3. The eigenvalue associated with -2 satisfies that $2v_1 + 2v_2 + v_3 = 0$, $2v_1 + 2v_2 + v_3 = 0$ and $v_1 + v_2 + 3v_3 = 0$. Subtracting twice the last equation from the first or second, we see that $-5v_3 = 0$ and so $v_3 = 0$. Then we $v_1 = -v_2$, so $(1, -1, 0)$ is an eigenvector associated with the eigenvalue -2.

3. (5 points) Find the solution to the initial value problem $y'' + y' - 2y = 3e^t$ where $y(0) = 0$ and $y'(0) = 0$.

 Name:__________________ Points earned:______

Solution: The characteristic equation for the associated homogeneous solution is $r^2 + r - 2 = 0$ which has roots $r = 1, -2$. Thus when using undetermined coefficients, the guess should have the form Ate^t. Which has first derivative $A(t+1)e^t$ and second derivative $A(t+2)e^t$. Substituting we want A so that

$$A(t+2)e^t + A(t+1)e^t - 2Ate^t = 3e^t$$

which yields $A = 1$. Thus the general form of the solution is $c_1e^{-2t} + c_2e^t + te^t$, and this has first derivative $-2c_1e^{-2t} + c_2e^t + (t+1)e^t$. Using the initial conditions we need $c_1 + c_2 = 0$ and $-2c_1 + c_2 + 1 = 0$. So $c_1 = \frac{1}{3}$ and $c_2 = -\frac{1}{3}$, given a solution of $\frac{1}{3}e^{-2t} + (t - \frac{1}{3})e^t$.

4. (5 points) Find a general solution to the following system of differential equations by transforming into a single differential equation,

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= -4x_1 + 3x_2. \end{aligned}$$

Solution: Using the first equation to solve for x_2 we get $x_2 = x_1' + 2x_1$. Substituting into the second equation we get $x_1'' + 2x_1' = -4x_1 + 3x_1' + 6x_1$, or $x_1'' - x_1' - 2x_1 = 0$ whose characteristic equation has roots $-1, 2$ so $x_1 = c_1e^{-t} + c_2e^{2t}$. Then $x_2 = -c_1e^{-t} + 2c_2e^{2t} + 2c_1e^{-t} + 2c_2e^{2t} = c_1e^{-t} + 4c_2e^{2t}$.

5. (5 points) Find the inverse of

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix},$$

without using a formula.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{bmatrix} \text{(row 2 minus 4 times row 1)}$$

$$\begin{bmatrix} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{bmatrix} \text{(row 1 minus 2 times row 2)}$$

So the inverse is

$$\begin{bmatrix} -9 & -2 \\ 1 & -4 \end{bmatrix}$$

6. (5 points) Given that t^5 is a solution to $t^2y'' - 6ty' + 10y = 0$ find another solution.

Solution: We use reduction of order, and guess $y = v(t)t^5$ is a solution. Then $y' = v't^5 + 5t^4 v$ and $y'' = v''t^5 + 10t^4 v' + 20t^3 v$. Substituting we get

$$t^2(v''t^5 + 10t^4 v' + 20t^3 v) - 6t(v't^5 + 5t^4 v) + 10t^5 v = 0$$

which simplifies to

$$t^7 v'' + 4t^6 v' = 0.$$

Rearranging we have that

$$\frac{v''}{v'} = \frac{-4}{t}.$$

Thus $\ln|v'| = -4\ln|t| + C = \ln(t^{-4}) + C$ and $v' = Ct^{-4}$. Thus $v = c_1 t^{-3} + c_2$, and t^2 is another solution.