1. (5 points) If y is a solution to $y'' + \sin(t)y' + t^2 y = \cos(t)$ such that $y(\pi) = 1$ and $y'(\pi) = 0$, find $y''(\pi)$ and $y'''(\pi)$.

Solution: First we note that $y''(\pi)+\sin(\pi)y'(\pi)+\pi^2 y(\pi) = \cos(\pi)$ and so $y''(\pi) = -1-\pi^2$. Differentiating with respect to t, we have that

$$y''' + \cos(t)y' + \sin(t)y'' + 2ty + t^2 y' = -\sin(t).$$

Thus $y'''(\pi) + \sin(\pi)y''(\pi) + (\cos(\pi) + \pi^2)y'(\pi) + 2\pi y(\pi) = -\sin(\pi)$ and hence $y'''(\pi) = 2\pi$.

2. (5 points) The matrix $A = \begin{bmatrix} -1 & -2 \\ 5 & 5 \end{bmatrix}$ has eigenvalues $2 \pm i$ and eigenvectors $\begin{bmatrix} -3 \pm i \\ 5 \end{bmatrix}$. Find two real valued fundamental solutions to $\mathbf{x}' = A\mathbf{x}$ and plot a phase portrait for the solutions. Be sure to label the two fundamental solutions.

Solution: From the eigenvalues and eigenvectors we know that the fundamental solutions are

$$e^{2t}\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t)\right) \quad \text{and} \quad e^{2t}\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t)\right)$$

Since the real part of the eigenvalues is positive, they spiral out. Oberserving that at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the derivative vector is $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ we know that it is spiraling counter-clockwise.

3. (5 points) The system of differential equations $\mathbf{x}' = A\mathbf{x}$ has a fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^t & e^{2t} \\ -e^t & e^{2t} \end{bmatrix}.$$

Find the general solution to

$$\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^t \sin(t) + \frac{e^{2t}}{t} \\ -e^t \sin(t) + frace^{2t}t \end{bmatrix}.$$

Solution: Let $g(t)$ be the non-homogeneous part of the differential equation. We will use variation of parameters to guess the solution, so assume $\mathbf{x} = \Psi\mathbf{v}(t)$. Thus $\Psi'\mathbf{v}(t)+\Psi\mathbf{v}'(t) = A\Psi\mathbf{v}(t) + g(t)$. Since Ψ is a solution to the associated homogeneous matrix differential equation, $A\Psi = \Psi'$ and so we have $\Psi\mathbf{v}'(t) = g(t)$.

$$\begin{bmatrix} e^t & e^{2t} & e^t \sin(t) + \frac{e^{2t}}{t} \\ -e^t & e^{2t} & -e^t \sin(t) + \frac{e^{2t}}{t} \end{bmatrix} \to \begin{bmatrix} e^t & e^{2t} & e^t \sin(t) + \frac{e^{2t}}{t} \\ 0 & 2e^{2t} & 2\frac{e^{2t}}{t} \end{bmatrix}$$

Thus $v_2'(t) = \frac{1}{t}$ and $e^t v_1'(t) + e^{2t} v_2'(t) = e^t \sin(t) + \frac{e^{2t}}{t}$. Thus $v_1'(t) = \sin(t)$ and the solution is

$$\begin{bmatrix} e^t & e^{2t} \\ -e^t & e^{2t} \end{bmatrix} \begin{bmatrix} -\cos(t) + c_1 \\ \ln|t| \end{bmatrix}.$$

4. (5 points) If y satisfies

$$y'' + 5y' + 6y = \cos(t) + t^2 \quad y(0) = 2 \quad y'(0) = -1$$

find $\mathcal{L}\{y\}$ and from this indicate what terms you would expect in the general solution. Be sure to tie your conclusions to the Laplace transform.

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$, then taking the Laplace transform

$$s^2 Y(s) - sy(0) - y'(0) + 5(Y(s) - y(0)) + 6Y(s) = \frac{s}{s^2+1} + \frac{2}{s^3}.$$

Thus

$$(s^2 + 5s + 6)Y(s) - 2s + 1 - 10 = \frac{s}{s^2+1} + \frac{2}{s^3}$$

and $Y(s) = \frac{s}{(s^2+1)(s^2+5s+6)} + \frac{2}{s^2(s^2+5s+6)} + \frac{2s+9}{s^2+5s+6}$. The partial fraction decomposition of $Y(s)$ will take the form $\frac{As+B}{s^2+1} + \frac{C+Ds+Es^2}{s^3} + \frac{F}{s+2} + \frac{G}{s+3}$, thus the expected form of the solution will be $A\cos(t) + B\sin(t) + \frac{C}{2}t^2 + Dt + E + Fe^{-2t} + Ge^{-3t}$.

5. (5 points) A model rocket has two engines that produce 200 $\mathrm{kg\cdot m/s^2}$ of thrust each for 20 seconds. Unfortunately one of the engines fails after 10 seconds. The constant for force exerted by air resistance is 10 $\mathrm{kg/s}$ and the rocket weighs 2 kg. For ease of calculation, treat the acceleration due to gravity as a constant 10 $\mathrm{m/s^2}$. Recall from the first exam that the differential equation governing the flight of the rocket is $y'' + 5y' = \frac{F-10}{10}$ where F is the force provided by the thrust provided by the engines. Using Laplace transforms, find the position of the rocket as a function of time.

Solution: First note that for the first 10 seconds $F = 400$, then for the next 10 seconds $F = 200$, and there after $F = 0$. Thus $F = 400 - 200u_{10}(t) - 200u_{20}(t)$ and so the differential equation is

$$y'' + 5y' = 39 - 20u_{10}(t) - 20u_{20}(t),$$

with $y(0) = y'(0) = 0$. Thus $(s^2 + 5s)\mathcal{L}y = \frac{39 - 10e^{-10s} - 10e^{-20s}}{s}$. Let $H(s) = \frac{1}{s^2(s+5)} = -\frac{1}{25s} + \frac{1}{5s^2} + \frac{1}{25}\frac{1}{s+5}$. Then $\mathcal{L}^{-1}\{H(s)\} = \frac{1}{25}\left(-1 + 5t + e^{-5t}\right)$ and $y(t) = \frac{39}{25}(-1 + 5t + e^{-5t}) - \frac{2}{5}u_{10}(t)\left(-1 + 5(t-10) + e^{-5(t-10)}\right) - frac25u_{20}(t)\left(-1 + 5(t-20) + e^{-5(t-20)}\right)$

6. (5 points) The matrix

$$A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$$

has eigenvalue 5 with algebraic multiplicity 2 and eigenvector $\begin{bmatrix}1\\2\end{bmatrix}$. Find the general form of the solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

Solution: We need to solve $(A - 5I)\nu = \begin{bmatrix}1\\2\end{bmatrix}$.

$$\begin{bmatrix}2 & -1 & 1\\4 & -2 & 2\end{bmatrix} \to \begin{bmatrix}2 & -1 & 1\\0 & 0 & 0\end{bmatrix}$$

Thus the solutions are $\begin{bmatrix}\frac{1+\alpha}{2}\\\alpha\end{bmatrix}$. But the part corresponding to α is a multiple of the eigenvector so the generic solution is

$$c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix} e^{5t} + \begin{bmatrix}1\\2\end{bmatrix} te^{5t} \right)$$