Do any three of the 4 problems

1. For systems with the potential energy function \( V(r) \) depending only on the distance \( r \), the wave function can be expressed as the product of the radial wave function \( R_{nl}(r) \) and the spherical harmonics \( Y_{lm}(\theta, \phi) \) where \( Y_{lm}(\theta, \phi) \) is the common eigenfunction of operators \( L^2 \) and \( L_z \) such that \( L^2 Y_{lm} = l(l+1) \hbar^2 Y_{lm} \) and \( L_z Y_{lm} = m \hbar Y_{lm} \). (a) Show that the radial wave equation is given by
   \[
   \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR_{nl}}{dr} \right) + \left( \frac{2m}{\hbar^2} [E - V(r)] - \frac{l(l+1)}{r^2} \right) R_{nl} = 0.
   \]
   (b) By making the substitution \( R_{nl} = u_{nl}(r)/r \), show that the radial wave function can be reduced to an effective one-dimensional Schrödinger equation
   \[
   d^2 u_{nl}/dr^2 + \left( \frac{2m}{\hbar^2} [E - V_{eff}] \right) u_{nl} = 0, \text{ where } V_{eff} = V(r) + \frac{l(l+1)}{2mr^2}.
   \]
   (c) For the case of the isotropic harmonic oscillator with \( V(r) = m\omega^2 r^2/2 \), the effective one-dimensional radial wave equation can be written as
   \[
   d^2 u_{nl}/d\rho^2 + \left( C - \rho^2 - l(l+1)/\rho^2 \right) u_{nl} = 0
   \]
   with the introduction of \( \rho = cr \), \( \alpha = (m\omega/\hbar)^2 \), and \( C = 2E/\hbar\omega \). By examining its asymptotic solutions at \( r \to 0 \) and \( r \to \infty \) respectively, show that \( u_{nl} \) can be written as \( u_{nl}(\rho) = \rho^{(l+1)/2} e^{-\rho/2} f(\rho) \) with \( f(\rho) \) satisfying the differential equation
   \[
   \rho(d^2 f/df^2) + 2(l+1-\rho^2)(df/d\rho) - (2l+3-C)pf = 0.
   \]
   (d) Making a change of variable \( \xi = \rho^2 \), show that the differential equation for \( f(\xi) \) is
   \[
   \xi(d^2 f/df^2) + (l+3/2-\xi)(df/d\xi) - (1/4)(2l+3-C)f = 0.
   \]
   This is in the form of the well-known differential equation \( xF''+(b-x)F'-aF = 0 \) whose solution is the confluent hypergeometric series
   \[
   F(a,b,x) = \sum \frac{\Gamma(a+s)\Gamma(b)x^s}{\Gamma(a)\Gamma(b+s)\Gamma(s+1)} = 1 + \frac{ax}{b} + \frac{a(a+1)x^2}{b(b+1)2!} + \ldots.
   \]
   It can be seen that \( F \) behaves as \( e^x \) for large \( x \). Show that the requirement that \( u_{nl} \) must be normalizable leads to \( E = (n+3/2)\hbar\omega \) where \( n = 2s+l \). Thus each energy state \( n \) is associated with several orbital angular momentum states \( l \) according to \( l=n-2s=n,n-2,...,(l \text{ or 0}) \).

2. A general angular momentum operator \( \vec{J} \) can be defined by the commutation relations of its components: \([J_x,J_y]=i\hbar J_z\). (a) Show that \([\vec{J},J_z]=0\). (b) Let the common eigenvector of \( \vec{J} \) and \( J_z \) be \( |\lambda, m\rangle \) such that \( \vec{J} |\lambda, m\rangle = \lambda \hbar \hat{\lambda} |\lambda, m\rangle \) and \( J_z |\lambda, m\rangle = m \hbar |\lambda, m\rangle \). Show that \( \lambda = f(j+1) \) and, for a given \( j, \ m = -j,-j+1,...,j \). (c) Show that possible values of \( j \) are 0,1/2,1,3/2,..., (d) Obtain the matrices \( J_x, J_y, \) and \( J_z \) in the representation of common eigenvectors of \( \vec{J} \) and \( J_z \) for \( j=3/2 \).
3. Consider a particle in a cylindrical box of radius $a$ and length $L$. Show, using cylindrical coordinates, that the possible values of the energy are

$$E = \left( \frac{\hbar^2}{2m} \right) \left( \frac{n \pi}{L} \right)^2 + \left( \epsilon_{m \pi} / a \right)^2$$

while the corresponding eigenfunctions are

$$\varphi_{nm\nu}(r) = NJ_{|\nu|}^{1/2} \frac{r}{a} \sin(n \pi r / L)$$

with $m = 0, \pm 1, \pm 2, \ldots$, $\nu = 1, 2, 3, \ldots$, and $\epsilon_{m \pi}$ being the $\nu$th root of the Bessel function of order $|\nu|$.

(Hints: \(\nabla^2 = \left(1/\rho^2\right) \left(\partial^2 / \partial \rho^2\right) + \left(\partial^2 / \partial \phi^2\right) + \left(\partial^2 / \partial z^2\right)\) in the cylindrical coordinates; the Bessel function of order $n$ satisfies the differential equation \(d^2 J_n / dr^2 + (1/r) dJ_n / dr + (1 - n^2 / r^2) J_n = 0\).)

4. The Green’s function $G(r)$ for a free particle is defined as the solution to the equation \(\left( \hbar^2 / 2m \right) \left(\nabla^2 + k^2\right) G(r) = \delta(r)\). (a) Using $G(\vec{r}) = (2\pi)^{-3} \int G(\vec{q}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{q}$ and $\delta(\vec{r}) = (2\pi)^{-3} \int e^{i\vec{q} \cdot \vec{r}} d^3\vec{q}$, show that $G(\vec{q}) = (2m/\hbar^2) (k^2 - \vec{q}^2)^{-1}$. (b) Determine $G(\vec{r})$. 
Mathematical Physics Qualifying Exam Name_____________

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1. For systems with the potential energy function \( V(r) \) depending only on the distance \( r \), the wave function can be expressed as the product of the radial wave function \( R_{nl}(r) \) and the spherical harmonics \( Y_{lm}(\theta, \phi) \) where \( Y_{lm}(\theta, \phi) \) is the common eigenfunction of operators \( L^2 \) and \( L_z \) such that \( L^2 Y_{lm} = l(l+1) \hbar^2 Y_{lm} \) and \( L_z Y_{lm} = m \hbar Y_{lm} \). (a) Show that the radial wave equation is given by
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with the introduction of \( \rho = \alpha r, \alpha = (m\omega/\hbar)^{1/2} \), and \( C = 2E/\hbar\omega \). By examining its asymptotic solutions at \( r \to 0 \) and \( r \to \infty \) respectively, show that \( u_{nl} \) can be written as \( u_{nl}(\rho) = \rho^{l+1/2}e^{-\rho^2/2}f(\rho) \) with \( f(\rho) \) satisfying the differential equation
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This is in the form of the well-known differential equation \( xF'' + (b-x)F' - aF = 0 \) whose solution is the confluent hypergeometric series
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F(a, b, x) = \sum_x \frac{\Gamma(a+s)\Gamma(b)x^s}{\Gamma(a)\Gamma(b+s)\Gamma(s+1)} = 1 + \frac{ax}{b} + \frac{a(a+1)x^2}{b(b+1)2!} + \ldots.
\]
It can be seen that \( F \) behaves as \( e^x \) for large \( x \). Show that the requirement that \( u_{nl} \) must be normalizable leads to \( E = (n+3/2)\hbar\omega \) where \( n = 2s+l \). Thus each energy state \( n \) is associated with several orbital angular momentum states \( l \) according to \( l = n-2s = n, n-2, \ldots, (l+1) \) or 0).

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$$\varphi_{nm}(r) = NJ_{nm}(\varepsilon_{nm} r/a)e^{im\phi} \sin(n\pi z/L)$$

with $m=0, \pm 1, \pm 2, \ldots$, $n = 1, 2, 3 \ldots$, and $\varepsilon_{nm}$ being the $n$th root of the Bessel function of order $n$.

(Hints: $\nabla^2 = (1/r^2) \left\{ (\partial/\partial r)[r^2 (\partial/\partial r)] + (\partial^2/\partial \phi^2) \right\} + (\partial^2/\partial z^2)$ in the cylindrical coordinates; the Bessel function of order $n$ satisfies the differential equation $d^2 J_n/dr^2 + (1/r)(dJ_n/dr) + (1 - n^2/r^2)J_n = 0$.)

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