Direction: This test is worth 200 points. You are required to complete this test within 75 minutes. In order to receive full credit, answer each problem completely and must show all work.

Please make sure that you have all the 11 pages. GOOD LUCK!
1. (20 points) Given \( \theta \), the random variable \( X \) has a binomial distribution with \( n = 3 \) and probability of success \( \theta \). If the prior density of \( \theta \) is

\[
h(\theta) = \begin{cases} 
2 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{otherwise},
\end{cases}
\]

what is the Bayes’ estimate of \( \theta \) for an absolute difference error loss if the sample consists of one observation \( x = 3 \)?

**Answer:** Since, the prior density of \( \theta \) is

\[
h(\theta) = \begin{cases} 
2 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{otherwise},
\end{cases}
\]

and the population density is

\[
f(x/\theta) = \binom{3}{x} \theta^x (1 - \theta)^{3-x},
\]

the joint density of the sample and the parameter is given by

\[
 u(3, \theta) = h(\theta) \cdot f(3/\theta) = 2 \theta^3,
\]

where \( \frac{1}{2} < \theta < 1 \). The marginal density of the sample (at \( x = 3 \)) is given by

\[
 g(3) = \int_{\frac{1}{2}}^{1} u(3, \theta) \, d\theta \\
 = \int_{\frac{1}{2}}^{1} 2 \theta^3 \, d\theta \\
 = \left[ \frac{\theta^4}{4} \right]_{\frac{1}{2}}^{1} \\
 = \frac{15}{32}.
\]

Therefore, the conditional density of \( \theta \) given \( X = 3 \) is

\[
k(\theta/x = 3) = \frac{u(3, \theta)}{g(3)} = \begin{cases} 
\frac{64}{15} \theta^3 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{elsewhere}.
\end{cases}
\]

Since, the loss function is absolute error, the Bayes’ estimator is the median of the probability density function \( k(\theta/x = 3) \). That is

\[
\frac{1}{2} = \int_{\frac{1}{2}}^{\hat{\theta}} \frac{64}{15} \theta^3 \, d\theta \\
= \frac{64}{15} \left[ \frac{\theta^4}{4} \right]_{\frac{1}{2}}^{\hat{\theta}} \\
= \frac{64}{60} \left[ \left( \frac{\hat{\theta}}{2} \right)^4 - \frac{1}{16} \right].
\]

Solving the above equation for \( \hat{\theta} \), we get

\[
\hat{\theta} = \sqrt[4]{\frac{17}{32}} = 0.8537.
\]
2. (20 points) Given $\theta$, the random variable $X$ has a binomial distribution with $n = 3$ and probability of success $\theta$. If the prior density of $\theta$ is

$$h(\theta) = \begin{cases} 
2 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{otherwise},
\end{cases}$$

what is the Bayes’ estimate of $\theta$ for a quadratic error loss if the sample consists of one observation $x = 3$?

**Answer:** Since, the prior density of $\theta$ is

$$h(\theta) = \begin{cases} 
2 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{otherwise},
\end{cases}$$

and the population density is

$$f(x/\theta) = \binom{3}{x} \theta^x (1 - \theta)^{3-x},$$

the joint density of the sample and the parameter is given by

$$u(3, \theta) = h(\theta) f(3/\theta) = 2 \theta^3,$$

where $\frac{1}{2} < \theta < 1$. The marginal density of the sample (at $x = 3$) is given by

$$g(3) = \int_{\frac{1}{2}}^{1} u(3, \theta) d\theta = \int_{\frac{1}{2}}^{1} 2 \theta^3 d\theta = \left[ \frac{\theta^4}{2} \right]_{\frac{1}{2}}^{1} = 15 \frac{1}{32}.$$ 

Therefore, the conditional density of $\theta$ given $X = 3$ is

$$k(\theta/x = 3) = \frac{u(3, \theta)}{g(3)} = \begin{cases} 
\frac{64}{15} \theta^3 & \text{if } \frac{1}{2} < \theta < 1 \\
0 & \text{elsewhere}.
\end{cases}$$

Since, the loss function is quadratic error, the Bayes’ estimator is the mean of the probability density function $k(\theta/x = 3)$. That is

$$\hat{\theta} = \int_{\frac{1}{2}}^{1} \frac{64}{15} \theta^3 d\theta$$

$$= \frac{64}{15} \left[ \frac{\theta^4}{4} \right]_{\frac{1}{2}}^{1}$$

$$= \frac{64}{75} \left[ 1 - \frac{1}{32} \right] = \frac{64}{75} \cdot \frac{31}{32} = \frac{62}{75}. $$
3. (20 points) Let $X_1, X_2, \ldots, X_5$ be a sample of size 5 from the uniform distribution on the interval $(0, \theta)$, where $\theta$ is unknown. Let the estimator of $\theta$ be $X_{\text{max}}$, where $X_{\text{max}}$ is the largest observation. Is $X_{\text{max}}$ an unbiased estimator of $\theta$?

**Answer:** The probability density function of $X_{\text{max}}$ is given by

$$g(x) = \frac{5!}{4!0!} [F(x)]^4 f(x)$$

$$= 5 \left(\frac{x}{\theta}\right)^4 \frac{1}{\theta}$$

$$= \frac{5}{\theta^5} x^4.$$

Since

$$E(X_{\text{max}}) = \int_0^\theta x g(x) \, dx$$

$$= \int_0^\theta \frac{5}{\theta^5} x^5 \, dx$$

$$= \frac{5}{6} \theta$$

$$\neq \theta,$$

therefore $X_{\text{max}}$ is not a unbiased estimator of $\theta$. 


4. (20 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with mean $\mu$ and variance $\sigma^2 > 0$. The maximum likelihood estimator of $\sigma^2$ is given by $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

Is this maximum likelihood estimator $\hat{\sigma}^2$ an unbiased estimator of the parameter $\sigma^2$?

Answer: In Example 15.13, we have shown that the maximum likelihood estimator of $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$ 

Now, we examine the unbiasedness of this estimator

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]$$

$$= E\left[\frac{n - 1}{n} \frac{1}{n - 1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]$$

$$= \frac{n - 1}{n} E\left[\frac{1}{n - 1} \sum_{i=1}^{n} (X_i - \bar{X})^2\right]$$

$$= \frac{n - 1}{n} E[\sigma^2]$$

$$= \frac{\sigma^2}{n} E\left[\frac{n - 1}{\sigma^2} S^2\right]$$

(since $\frac{n - 1}{\sigma^2} S^2 \sim \chi^2(n - 1)$)

$$= \frac{\sigma^2}{n} E[\chi^2(n - 1)]$$

$$= \frac{\sigma^2}{n} (n - 1)$$

$$= \frac{n - 1}{n} \sigma^2$$

$$\neq \sigma^2.$$ 

Therefore, the maximum likelihood estimator of $\sigma^2$ is a biased estimator.
5. (20 points) Let $X_1, X_2, \ldots, X_7$ be a random sample of size 7 from a distribution with $-\infty < \mu < \infty$, and variance $\sigma^2 > 0$. The statistics $\bar{X}$ and $Y = \frac{X_1 + 2X_2 + \cdots + 7X_7}{28}$ are both unbiased estimators of $\mu$. Which of the two estimators of $\mu$ is an efficient unbiased estimator of $\mu$?

**Answer:** We determine the variance of both the estimators. The variances of these estimators are given by

$$V ar(\bar{X}) = V ar\left(\frac{X_1 + X_2 + \cdots + X_7}{7}\right)$$

$$= \frac{1}{49} [V ar(X_1) + V ar(X_2) + \cdots + V ar(X_7)]$$

$$= \frac{1}{49} 7\sigma^2$$

$$= \frac{1}{7} \sigma^2$$

$$= \frac{28}{196} \sigma^2$$

and

$$V ar(Y) = V ar\left(\frac{X_1 + 2X_2 + \cdots + 7X_7}{6}\right)$$

$$= \frac{1}{(28)^2} [V ar(X_1) + 4V ar(X_2) + \cdots + 49V ar(X_7)]$$

$$= \frac{1}{784} 140 \sigma^2$$

$$= \frac{5}{28} \sigma^2$$

$$= \frac{35}{196} \sigma^2.$$ 

Therefore

$$\frac{28}{196} \sigma^2 = V ar(\bar{X}) < V ar(Y) = \frac{35}{196} \sigma^2.$$ 

Hence, $\bar{X}$ is more efficient than the estimator $Y$. 

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6. (20 points) Let \( T_1 \) and \( T_2 \) be estimators of a population parameter \( \theta \) based upon the same random sample. If \( T_i \sim N(\theta, \sigma_i^2) \) for \( i = 1, 2 \) and if \( T = bT_1 + (1 - b)T_2 \), then for what value of \( b \), \( T \) is a minimum variance unbiased estimator of \( \theta \)?

**Answer:** The variance of \( T \) is given by

\[
Var(T) = Var(bT_1 + (1 - b)T_2)
\]

\[
= b^2 Var(T_1) + (1 - 2b + b^2) Var(T_2) + 2(b - b^2) Cov(T_1, T_2)
\]

\[
= b^2 \sigma_1^2 + (1 - 2b + b^2) \sigma_2^2 + 2(b - b^2) Cov(T_1, T_2).
\]

Differentiating \( Var(T) \) with respect to \( b \), we get

\[
\frac{d}{db} Var(T) = 2b \sigma_1^2 - 2 \sigma_2^2 + 2(b - b^2) \sigma_2^2 + 2 Cov(T_1, T_2) - 4b Cov(T_1, T_2).
\]

Equating this derivative to zero and solving for \( b \), we get

\[
b = \frac{\sigma_2^2 - 2 Cov(T_1, T_2)}{\sigma_1^2 + \sigma_2^2 - 2 Cov(T_1, T_2)}.
\]
7. (20 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with unknown mean $\mu$ and known variance $\sigma^2 > 0$. What is the maximum likelihood estimator of $\mu$? Is this maximum likelihood estimator an efficient estimator of $\mu$?

**Answer:** The probability density function of the population is

$$f(x; \mu) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} (x-\mu)^2}.$$  

Thus

$$\ln f(x; \mu) = -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} (x - \mu)^2$$

and hence

$$\ln L(\mu) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2.$$  

Taking the derivative of $\ln L(\mu)$ with respect to $\mu$, we get

$$\frac{d\ln L(\mu)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu).$$

Setting this derivative to zero and solving for $\mu$, we see that $\hat{\mu} = \bar{X}$.

The variance of $\bar{X}$ is given by

$$\text{Var} (\bar{X}) = \text{Var} \left( \frac{X_1 + X_2 + \cdots + X_n}{n} \right)$$

$$= \frac{\sigma^2}{n}.$$  

Next we determine the Cramér-Rao lower bound for the estimator $\bar{X}$. We already know that

$$\frac{d\ln L(\mu)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)$$

and hence

$$\frac{d^2 \ln L(\mu)}{d\mu^2} = -\frac{n}{\sigma^2}.$$  

Therefore

$$E \left( \frac{d^2 \ln L(\mu)}{d\mu^2} \right) = -\frac{n}{\sigma^2}$$

and

$$-\frac{1}{E \left( \frac{d^2 \ln L(\mu)}{d\mu^2} \right)} = \frac{\sigma^2}{n}.$$  

Thus

$$\text{Var} (\bar{X}) = -\frac{1}{E \left( \frac{d^2 \ln L(\mu)}{d\mu^2} \right)}$$

and $\bar{X}$ is an efficient estimator of $\mu$. Since every efficient estimator is a uniform minimum variance unbiased estimator, therefore $\bar{X}$ is a uniform minimum variance unbiased estimator of $\mu$. 
8. (20 points) If $X_1, X_2, ..., X_n$ is a random sample from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \theta^x (1 - \theta)^{1-x} & \text{if } x = 0,1 \\ 0 & \text{elsewhere} , \end{cases}$$

where $0 < \theta < 1$. Find a sufficient statistic for $\theta$.

**Answer:** Since

$$f(x; \theta) = e^{x \ln \theta + (1-x) \ln(1-\theta)}$$

$$= e^{x \ln \left( \frac{\theta}{\theta + (1-\theta)} \right) + \ln(1-\theta)}$$

we have

$$k(x) = x$$

and hence a sufficient statistics for $\theta$ is given by

$$\sum_{i=1}^{n} k(X_i) = \sum_{i=1}^{n} X_i.$$
9. (20 points) Let $X_1, X_2, ..., X_n$ be a random sample from a population $X$ with density function

$$ f(x; \theta) = \begin{cases} \theta \alpha x^{\alpha-1}e^{-\theta x^\alpha} & \text{for } 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases} $$

where $\theta > 0$ and $\alpha > 0$ are parameters. What is a sufficient statistic for the parameter $\theta$ for a fixed $\alpha$?

**Answer:** Since

$$ f(x; \theta) = e^{\ln \theta + \ln \alpha + (\alpha - 1) \ln x - \theta x^\alpha} $$

we have

$$ k(x) = x^\alpha $$

and hence a sufficient statistics for $\theta$ for a fixed $\alpha$ is given by

$$ \sum_{i=1}^{n} k(X_i) = \sum_{i=1}^{n} X_i^\alpha. $$
10. (20 points) Define the following terminologies.

(a) statistic

Answer: Let $X_1, X_2, ..., X_n$ be a random sample from a population with density $f(x; \theta)$, where $\theta$ is an unknown parameter. A statistic is a function of the sample $X_1, X_2, ..., X_n$ which is free of the parameter $\theta$.

(b) estimator of a parameter

Answer: Let $X \sim f(x; \theta)$ and $X_1, X_2, ..., X_n$ be a random sample from the population $X$. Any statistic that can be used to guess the parameter $\theta$ is called an estimator of $\theta$.

(c) unbiased estimator of a parameter

Answer: An estimator $\hat{\theta}$ of $\theta$ is said to be an unbiased estimator of $\theta$ if and only if $E(\hat{\theta}) = \theta$.

(d) minimum variance unbiased estimator

Answer: An unbiased estimator $\hat{\theta}$ of $\theta$ is said to be a uniform minimum variance unbiased estimator of $\theta$ if and only if

$$\text{Var}(\hat{\theta}) \leq \text{Var}(\hat{T})$$

for any unbiased estimator $\hat{T}$ of $\theta$.

(e) efficient estimator of a parameter

Answer: An unbiased estimator $\hat{\theta}$ is called an efficient estimator if it satisfies Cramér-Rao lower bound, that is

$$\text{Var}(\hat{\theta}) = \frac{1}{E \left[ \left( \frac{\partial \ln L(\theta)}{\partial \theta} \right)^2 \right]}.$$

(f) sufficient estimator of a parameter

Answer: An estimator $\hat{\theta}$ of the parameter $\theta$ is said to be a sufficient estimator of $\theta$ if the conditional distribution of the sample given the estimator $\hat{\theta}$ does not depend on the parameter $\theta$. 