1. Consider the group \( G = \{1, i, -1, -i\} \) under multiplication. For an arbitrary but fixed element \( a \) in \( G \), define a map \( h_a : G \rightarrow G \) by \( h_a(x) = xa \) for all \( x \in G \). Find the maps \( h_i \), \( h_1 \), \( h_{-1} \) and \( h_{-i} \). Show that they are permutations on the set of elements in \( G \).

**Answer:** Let \( G \) be \( \{1, i, -1, -i\} \). The Cayley table of \( G \) is the following:

\[
\begin{array}{cccc}
  \cdot & 1 & i & -1 & -i \\
 1 & 1 & i & -1 & -i \\
i & i & -1 & -i & 1 \\
-1 & -1 & -i & 1 & i \\
-i & -i & 1 & i & -1 \\
\end{array}
\]

Hence \( h_1 = \begin{pmatrix} 1 & i & -1 & -i \\ 1 & i & -1 & -i \end{pmatrix} \), \( h_i = \begin{pmatrix} 1 & i & -1 & -i \\ i & -1 & -i & 1 \end{pmatrix} \), \( h_{-1} = \begin{pmatrix} 1 & i & -1 & -i \\ -1 & -i & 1 & i \end{pmatrix} \), \( h_{-i} = \begin{pmatrix} 1 & i & -1 & -i \\ -i & 1 & i & -1 \end{pmatrix} \). Since these functions are one-to-one and onto, they are permutations on \( G \).

2. For each \( a \) in the arbitrary group \( G \), define a mapping \( h_a : G \rightarrow G \) by \( h_a(x) = xa \) for all \( x \) in \( G \). Prove that \( h_a \) is a permutation on the set of elements in \( G \).

**Answer:** For each \( a \in G \), define the map \( h_a : G \rightarrow G \) by \( h_a(x) = xa \) for all \( x \in G \). To show \( h_a \) is a permutation we have to show \( h_a \) is well-defined, one-to-one and onto. First we show \( h_a \) is well-defined. Suppose \( x = y \). Then \( xa = ya \). Hence \( h_a(x) = h_a(y) \). Therefore \( h_a \) is well-defined. Next, we show \( h_a \) is one-to-one. Suppose \( x \neq y \). Then \( xa \neq ya \). This implies that \( h_a(x) \neq h_a(y) \). So \( h_a \) is one-to-one. Finally, we show that \( h_a \) is onto. Let \( b \in G \) and find an element \( x \in G \) such that \( h_a(x) = b \). This implies that \( xa = b \) which yields \( x = ba^{-1} \). This shows that \( h_a \) is onto.

3. For each \( a \) in the arbitrary group \( G \), define a mapping \( h_a : G \rightarrow G \) by \( h_a(x) = xa \) for all \( x \). Show that the set of permutations \( H = \{ h_a \mid a \in G \} \) is a group under function composition.
**Answer:** Let $H = \{ h_a \mid a \in G \}$. We want to show that $H$ is a group under function composition. If $e$ is the identity of $G$, then $h_e$ is the identity of $H$. This can be seen from the followings: $h_e h_e(x) = h_e(xa) = h_a(ea) = xae = xae = h_a(x)$. So $h_e h_a = h_a$ for any $a \in G$. Similarly, we have $h_a h_e = h_a$ for every $a \in G$. Next, we show that if $a \in G$, then $h_{a^{-1}}$ is the inverse $h_a$. To see this consider $h_{a^{-1}} h_a(x) = h_{a^{-1}} (xa) = (xa)a^{-1} = x = xe = h_e(x)$. Hence $h_{a^{-1}} h_a = h_e$ and similarly $h_a h_{a^{-1}} = h_e$. Therefore $h_{a^{-1}}$ is the inverse $h_a$ in $H$. Since the function composition is associative, the associativity law holds in $H$. Thus $H$ is a group consisting of a set of all permutations on $G$.

4. For each $a$ in the arbitrary group $G$, define a mapping $h_a : G \to G$ by $h_a(x) = xa$ for all $x \in G$. Show that the permutation group $H = \{ h_a \mid a \in G \}$ is anti-isomorphic to the group $G$. [Note: A function $\phi : H \to G$ is an anti-isomorphism if and only if $\phi$ is one-to-one, onto, and satisfies $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in H$.]

**Answer:** We want to show $G$ is anti-isomorphic to $H$. That is, we want to produce an anti-isomorphism from $G$ to $H$. Consider the mapping $\phi : G \to H$ defined by $\phi(a) = h_a$ for all $a \in G$. First, we show $\phi$ is well-defined. Suppose $a = b$. This implies $xa = xb$ for $x \in G$. Hence from the definition of $h_a$, we get $h_a(x) = h_b(x)$ for all $x \in G$. This yields $h_a = h_b$ and therefore $\phi(a) = \phi(b)$. Hence we see that $\phi$ is well-defined. Second, we show that $\phi$ is one-to-one. Suppose $a \neq b$. Then $xa \neq xb$. Hence $h_a(x) \neq h_b(x)$ and thus $h_a \neq h_b$. Therefore $\phi(a) \neq \phi(b)$. That is $\phi$ is one-to-one. Third, we show that $\phi$ is onto. Since $\phi(a) = h_a$ for each $a \in G$, $\phi$ is clearly onto. Finally, we show that $\phi(ab) = \phi(b)\phi(a)$ for all $a, b \in G$. For this first we show $h_{ab} = h_b h_a$. Since $h_{ab}(x) = xab = (xa)b = h_b(xa) = h_b(h_a(x)) = h_b h_a(x)$, we see that $h_{ab} = h_b h_a$. Since $\phi(ab) = h_{ab} = h_b h_a = \phi(b)\phi(a)$, the mapping $\phi$ is an anti-isomorphism and $G$ and $H$ are anti-isomorphic to each other.

5. For each $a$ in an arbitrary group $G$, define a mapping $\phi_a(x) = axa^{-1}$ for all $x \in G$. Prove that $\phi_a$ is an isomorphism from $G$ onto $G$.

**Answer:** Define $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$ for all $x \in G$. First, we show $\phi_a$ is well-defined. Suppose $x = y$. This implies $ax = ay$ and $axa^{-1} = aya^{-1}$. Hence $\phi_a(x) = \phi_a(y)$. Therefore $\phi_a$ is well-defined. Second, we show $\phi_a$ is one-to-one. Suppose $x \neq y$. Then $axa^{-1} \neq aya^{-1}$ and hence $\phi_a(x) \neq \phi_a(y)$. So $\phi_a$ is one-to-one. Third, we show $\phi_a$ is onto. Pick an arbitrary element $b$ in $G$ and find an element $x$ in $G$ such that $\phi_a(x) = b$. If an
element $x$ exists, then $axa^{-1} = b$. Hence $x = a^{-1}ba$. Since $a^{-1}ba \in G$, such an element $x$ always exists. Thus $\phi_a$ is onto. Finally, we show $\phi_a$ preserves group operation. Consider $\phi_a(xy) = axya^{-1} = axa^{-1}aya^{-1} = \phi_a(x)\phi_a(y)$. Therefor $\phi_a$ is an isomorphism from $G$ onto $G$. [ Recall that $\phi_a$ is known as the inner automorphism of the group $G$.]

6. Let $G$ be a cyclic group of order 105. Find all subgroups of $G$.

**Answer:** Since $G$ is a cyclic group of order 105, therefore $G = \langle g \rangle$ for some $g \in G$, and $G \simeq \mathbb{Z}_{105}$. The divisors of 105 are 1, 3, 5, 7, 15, 21, 35, 105. Hence the subgroups of $G$ are: $\langle g^{105} \rangle, \langle g^{35} \rangle, \langle g^{21} \rangle, \langle g^{15} \rangle, \langle g^{7} \rangle, \langle g^{5} \rangle$ and $\langle g^{1} \rangle$. Note that $\langle g^{105} \rangle$ is same as $\langle e \rangle$.

7. Find an isomorphism from the group of integers $(\mathbb{Z}, +)$ to the group of even integers $(2\mathbb{Z}, +)$.

**Answer:** We want to find an isomorphism from $\phi : \mathbb{Z} \rightarrow 2\mathbb{Z}$. Define $\phi(x) = 2x$ for each $x \in \mathbb{Z}$. Then $\phi$ is one-to-one and also onto. Next show that $\phi(x + y) = \phi(x) + \phi(y)$ which is really easy. Hence $\mathbb{Z} \simeq 2\mathbb{Z}$.

8. Show that $U(8)$ is isomorphic to $U(12)$.

**Answer:** We want to show $U(8) \simeq U(12)$. Since $U(8) = \{1, 3, 5, 7\}$ and $U(12) = \{1, 5, 7, 11\}$ define a mapping $\phi : U(8) \rightarrow U(12)$ by

$$
\phi = \begin{pmatrix}
1 & 3 & 5 & 7 \\
1 & 5 & 7 & 11
\end{pmatrix}.
$$

Then clearly $\pi$ is one-to-one and onto. Next we show $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in U(8)$.

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We see that if we replace each entry $x$ in the multiplication table for $U(8)$ by $\phi(x)$, then we get the multiplication table for $U(12)$. Hence $\phi$ satisfies $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in U(8)$.  

9. Let $\mathbb{C}$ be the set of complex numbers and

$$M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Show that $\mathbb{C}$ and $M$ are isomorphic under addition.

**Answer:** We want to show that $\mathbb{C} \simeq M$. Define a function $\phi : \mathbb{C} \to M$ as

$$\phi(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{for all } a + ib \in \mathbb{C}.$$

First, we show that $\phi$ is well-defined. Suppose $a_1 + ib_1 = a_2 + ib_2$. Then we have $a_1 = a_2$ and $b_1 = b_2$. Therefore $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$ and $\phi(a_1 + ib_1) = \phi(a_2 + ib_2)$. Thus $\phi$ is well-defined. Second, we show $\phi$ is one-to-one. Suppose $\phi(a_1 + ib_1) = \phi(a_2 + ib_2)$. Then $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$. Therefore $a_1 = a_2$ and $b_1 = b_2$. This implies that $a_1 + ib_1 = a_2 + ib_2$. Hence $\phi$ is one-to-one. It is easy to see $\phi$ is onto. Finally, we show $\phi$ satisfies $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ for all $z_1, z_2 \in \mathbb{C}$. For $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ consider

$$\phi(z_1 + z_2) = \phi(a_1 + ib_1 + a_2 + ib_2)$$

$$= \phi(a_1 + a_2 + i(b_1 + b_2))$$

$$= \begin{pmatrix} a_1 + a_2 & -b_1 - b_2 \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$$

$$= \phi(a_1 + ib_1) + \phi(a_2 + ib_2)$$

$$= \phi(z_1) + \phi(z_2).$$

Therefore $\phi$ is an isomorphism from $\mathbb{C}$ onto $M$, and $\mathbb{C} \simeq M$. 